19.6 COMBINATORIAL NEURAL CODES

Neural codes are the brain's way of representing, transmitting, and storing information about the world. Combinatorial neural codes are based on binary patterns of neural activity, as opposed to the precise timing or rate of neural activity. The structure of a combinatorial code may reflect important aspects of the represented stimuli or network architecture. Combinatorial codes can be analyzed using an algebraic object called the neural ring.

19.6.1 BASIC CONCEPTS

From simultaneous recordings of neurons in the brain we can infer which subsets of neurons tend to fire together. This information is captured by a combinatorial code.

Definitions:

The set of neurons is denoted by $[n] = \{1, \ldots, n\}$.

An ***action potential***, or ***spike***, is an electrical event in a single neuron. This is the fundamental unit of neural activity. We say that a neuron "fires" action potentials, or spikes.

A ***spike train*** is a sequence of spike times for a single neuron. This captures the electrical activity of the neuron over time.

A ***codeword*** is a string of 0s and 1s, with a 1 for each active neuron and a 0 denoting silence; equivalently, it is a subset $\sigma \subseteq [n]$ of (active) neurons firing together. For example, if $n = 6$ the subset $\sigma = \{145\} \subseteq [6]$ is also denoted 100110.

A ***combinatorial neural code*** is a collection of codewords $\mathcal{C} \subseteq 2^{[n]}$. In other words, it is a binary code of length n, where each binary digit is interpreted as the "on" or "off" state of a neuron.

A ***maximal codeword*** is a codeword that is maximal in the code under inclusion. If $\sigma \in \mathcal{C}$ is maximal, then there is no $\tau \in \mathcal{C}$ such that $\tau \supsetneq \sigma$.

An ***abstract simplicial complex*** $\Delta \subseteq 2^{[n]}$ is a collection of subsets of $[n]$ that is closed under inclusion (see §15.6.1). That is, if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. A ***facet*** of Δ is an element of Δ that is maximal under inclusion.

The ***simplicial complex of a code***, $\Delta(\mathcal{C})$, is the smallest abstract simplicial complex on $[n]$ that contains all elements of $\mathcal{C}$:

$$\Delta(\mathcal{C}) = \{\sigma \subseteq [n] \mid \sigma \subseteq \tau \text{ for some } \tau \in \mathcal{C}\}.$$

Facts:

1. Spikes (action potentials) are all-or-none electrical events. It thus suffices to keep track only of the spike times, as in a spike train.

2. Most combinatorial neural codes appear ill-suited for error correction [CuEtal13a].

3. Simplicial complexes are heavily-studied objects in topology and algebraic combinatorics.

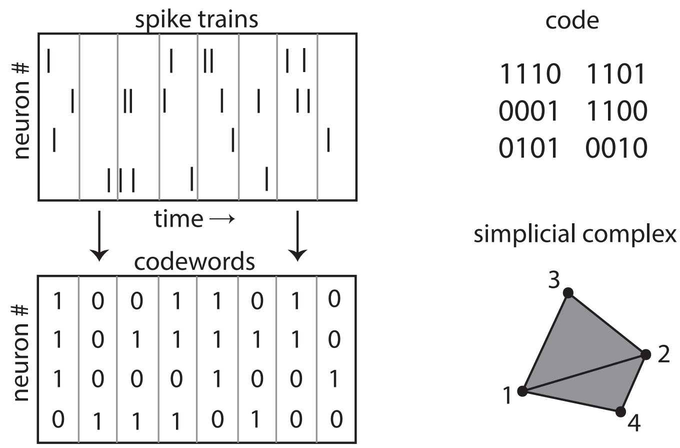
4. Each facet of $\Delta(\mathcal{C})$ corresponds to a maximal codeword of $\mathcal{C}$.

5. The simplicial complex $\Delta(\mathcal{C})$ is useful for analyzing a code, but discards important information. All codes with the same maximal codewords have the same simplicial complex.

6. Manin [Ma15] provides an historical overview contrasting neural codes with error-correcting codes and cryptography.

Example:

1. Combinatorial codes can be obtained from neural data by temporally binning the spikes into patterns of 0s and 1s. The following figure depicts a set of binned spike trains and the resulting codewords. The set of unique codewords is the code $\mathcal{C}$. The simplicial complex $\Delta(\mathcal{C})$ has facets corresponding to the two maximal codewords, 1110 and 1101.



19.6.2 THE CODE OF A COVER

An important type of combinatorial neural code is one defined by an arrangement of open sets in Euclidean space. The open sets correspond to receptive fields.

Definitions:

A ***stimulus space*** X is a parametric space of stimuli. The stimuli could be sensory, such as visual, auditory, or olfactory signals, or higher-level, such as an animal's position in space. Typically, a stimulus space is modeled as a subset of Euclidean space, $X \subseteq \mathcal{R}^d$.

A ***receptive field*** is a subset $U_i \subseteq X$ of the stimulus space corresponding to a single neuron i. The stimuli in U_i induce neuron i to fire.

A subset $V \subseteq \mathcal{R}^n$ is ***convex*** if, given any pair of points $x, y \in V$, the point $z = tx+(1-t)y$ is contained in V for any $t \in [0,1]$.

Convex receptive fields are convex subsets $U_i \subseteq X$.

A collection of open sets $\mathcal{U} = \{U_1, \ldots, U_n\}$ is an ***open cover*** of their union $\bigcup_{i=1}^n U_i$.

$\mathcal{U}$ is a ***good cover*** if every nonempty intersection $\bigcap_{i \in \sigma} U_i$ is contractible (that is, if it can be continuously shrunk to a point).

The ***nerve*** of an open cover $\mathcal{U}$ is the simplicial complex

$$\mathcal{N}(\mathcal{U}) = \{\sigma \subseteq [n] \mid \bigcap_{i \in \sigma} U_i \neq \emptyset\}.$$

Given an open cover $\mathcal{U}$, the ***code of the cover*** is the combinatorial neural code

$$\mathcal{C}(\mathcal{U}) = \{\sigma \subseteq [n] \mid \bigcap_{i \in \sigma} U_i \setminus \bigcup_{j \in [n]\setminus\sigma} U_j \neq \emptyset\}.$$

Facts:

1. Neurons in many brain areas, such as sensory cortices and the hippocampus, have activity patterns that can be characterized by receptive fields.

2. Receptive fields are computed experimentally by correlating neural responses to independently measured external stimuli.

3. Intersections of convex sets are always convex, and all convex sets are contractible. Thus, any open cover consisting of convex sets is a good cover.

4. Each codeword in $\mathcal{C}(\mathcal{U})$ corresponds to a region that is defined by the intersections of the open sets in $\mathcal{U}$ [CuEtal13b].

5. If $\mathcal{U}$ is an open cover, then $\mathcal{C}(\mathcal{U}) \subseteq \mathcal{N}(\mathcal{U})$ and $\Delta(\mathcal{C}(\mathcal{U})) = \mathcal{N}(U)$. The nerve of the cover can thus be recovered from the code by completing it to a simplicial complex, but the code contains additional information about $\mathcal{U}$ that is not captured by the nerve alone.

6. *Nerve lemma*: If $\mathcal{U}$ is a good cover, then the covered space $Y = \bigcup_{i=1}^{n} U_i$ is homotopy-equivalent to $\mathcal{N}(\mathcal{U})$. In particular, Y and $\mathcal{N}(\mathcal{U})$ have exactly the same homology groups.

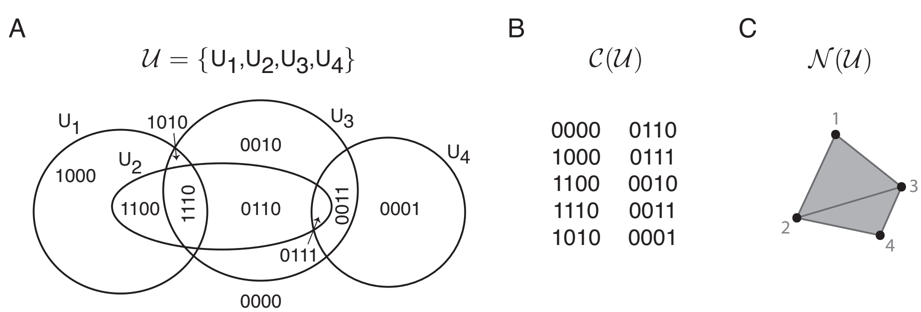
7. *Helly's theorem*: Consider k convex subsets $U_1, \ldots, U_k \subseteq \mathcal{R}^d$, for $d < k$. If the intersection of every $d+1$ of these sets is nonempty, then the full intersection $\bigcap_{i=1}^{k} U_i$ is also nonempty.

8. In addition to Helly's theorem and the Nerve lemma, there is a great deal known about $\mathcal{N}(\mathcal{U})$ for collections of convex sets in $\mathcal{R}^d$. In particular, the f-vectors of such simplicial complexes have been completely characterized by G. Kalai [Ka84], [Ka86].

9. The Nerve lemma has been exploited in the context of two-dimensional place field codes to show that topological features of an animal's environment could be inferred from neural codes representing position in the hippocampus [CuIt08].

Example:

1. The following figure, adapted from [CuEtal15], depicts an open cover consisting of four convex sets (A) as well as the corresponding code (B). The nerve of the cover (C) is identical to the simplicial complex $\Delta(\mathcal{C})$.



19.6.3 THE NEURAL RING AND IDEAL

The structure of a combinatorial code can be analyzed using the neural ring and ideal. These are algebraic objects that keep track of the combinatorics of the code, much as the Stanley-Reisner ring and ideal encode a simplicial complex [MiSt05]. For more details, see [CuEtal13b].

Definitions:

$\mathcal{F}_2$ is the field with two elements $\{0,1\}$. We can regard a codeword on n neurons as an element of $\mathcal{F}_2^n$ and a combinatorial neural code as a subset $\mathcal{C} \subseteq \mathcal{F}_2^n$.

$\mathcal{F}_2[x_1, \ldots, x_n]$ is a polynomial ring with coefficients in $\mathcal{F}_2$.

The ***ideal*** $\boldsymbol{I_{\mathcal{C}}}$ is the set of polynomials that vanish on all codewords in $\mathcal{C}$:

$$I_{\mathcal{C}} = I(\mathcal{C}) = \{f \in \mathcal{F}_2[x_1, \ldots, x_n] \mid f(c) = 0 \text{ for all } c \in \mathcal{C}\}.$$

The ***neural ring*** $\boldsymbol{R_{\mathcal{C}}}$ is the quotient ring

$$R_{\mathcal{C}} = \mathcal{F}_2[x_1, \ldots, x_n]/I_{\mathcal{C}}.$$

A ***pseudo-monomial*** is a polynomial $f \in \mathcal{F}_2[x_1, \ldots, x_n]$ that can be written as

$$f = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j),$$

where $\sigma, \tau \subseteq [n]$ satisfy $\sigma \cap \tau = \emptyset$.

For any binary string $v \in \mathcal{F}_2^n$, the ***indicator function***

$$\chi_v = \prod_{\{i \mid v_i = 1\}} x_i \prod_{\{j \mid v_j = 0\}} (1 - x_j)$$

is a pseudo-monomial with the property that $\chi_v(v) = 1$ and $\chi_v(c) = 0$ for any $c \neq v$.

The ***neural ideal*** $\boldsymbol{J_{\mathcal{C}}}$ is generated by the indicator functions of all non-codewords:

$$J_{\mathcal{C}} = \langle \chi_\nu \mid \nu \in \mathcal{F}_2^n \setminus \mathcal{C} \rangle.$$

A pseudo-monomial $f \in J_{\mathcal{C}}$ is called ***minimal*** if there does not exist another pseudo-monomial $g \in J_{\mathcal{C}}$ with $\deg(g) < \deg(f)$ such that $f = hg$ for some $h \in \mathcal{F}_2[x_1, \ldots, x_n]$.

The ***canonical form*** of $J_{\mathcal{C}}$ is the set of all minimal pseudo-monomials:

$$\mathrm{CF}(J_{\mathcal{C}}) = \{f \in J_{\mathcal{C}} \mid f \text{ is a minimal pseudo-monomial}\}.$$

Facts:

1. A polynomial $f \in \mathcal{F}_2[x_1, \ldots, x_n]$ can be evaluated on a binary string of length n (such as a codeword) by simply replacing each indeterminate x_i with the 0/1 value of the i^{th} position in the string. For example, if $f = x_1 x_3 (1 - x_2) \in \mathcal{F}_2[x_1, \ldots, x_4]$, then $f(1011) = 1$ and $f(1100) = 0$.

2. Irrespective of $\mathcal{C}$, the ideal $I_{\mathcal{C}}$ always contains the relations $\mathcal{B} = \langle x_1^2 - x_1, \ldots, x_n^2 - x_n \rangle$, due to the binary nature of codewords.

3. The ideals $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ carry all the combinatorial information about the code $\mathcal{C}$. They are closely related: $I_{\mathcal{C}} = J_{\mathcal{C}} + \mathcal{B}$.

4. *Fundamental lemma*: Let $\mathcal{C} \subseteq \{0,1\}^n$ be a neural code, and let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be any collection of open sets (not necessarily convex) such that $\mathcal{C} = \mathcal{C}(\mathcal{U})$. Then, for any pair of subsets $\sigma, \tau \subseteq [n]$,

$$\prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j) \in I_{\mathcal{C}} \;\Leftrightarrow\; \bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j.$$

5. The canonical form is a special basis, similar to a Grobner basis but tailored to a different purpose. From the canonical form one can read off minimal relationships between receptive fields.

6. The canonical form $\mathrm{CF}(J_{\mathcal{C}})$ can be computed algorithmically, starting from the code $\mathcal{C}$. In [CuEtal13b, Section 4.5], one such algorithm is described that uses the primary decomposition of pseudo-monomial ideals. This algorithm has since been improved [CuYo15], and software for computing $\mathrm{CF}(J_{\mathcal{C}})$ is publicly available at

- `https://github.com/nebneuron/neural-ideal`

Examples:

1. The code $\mathcal{C}(\mathcal{U})$ shown in panel (B) of §19.6.2, Example 1 has ten codewords and six non-codewords: $0100, 1001, 0101, 1101, 1011$, and 1111. The neural ideal is

$$\begin{aligned} J_{\mathcal{C}} \;=\; & \langle x_2(1-x_1)(1-x_3)(1-x_4),\; x_1x_4(1-x_2)(1-x_3),\; x_2x_4(1-x_1)(1-x_3), \\ & x_1x_2x_4(1-x_3),\; x_1x_3x_4(1-x_2),\; x_1x_2x_3x_4\rangle. \end{aligned}$$

The canonical form is

$$\mathrm{CF}(J_{\mathcal{C}}) = \{x_1x_4, x_2(1-x_1)(1-x_3),\, x_2x_4(1-x_3)\}.$$

Using the fundamental lemma, we can read off the following receptive field relationships: $U_1 \cap U_4 = \emptyset$, $U_2 \subseteq U_1 \cup U_3$, and $U_2 \cap U_4 \subseteq U_3$. This is consistent with the original arrangement of open sets shown in panel (A) of §19.6.2, Example 1.

2. The code $\mathcal{C} = \{111, 011, 001, 000\}$ on three neurons has the canonical form $\mathrm{CF}(J_{\mathcal{C}}) = \{x_1(1-x_2), x_1(1-x_3), x_2(1-x_3)\}$. This indicates that $U_1 \subseteq U_2$, $U_1 \subseteq U_3$, and $U_2 \subseteq U_3$.

19.6.4 CONVEX CODES

Definitions:

Let $\mathcal{C}$ be a combinatorial neural code on n neurons.

If there exists an open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ such that $\mathcal{C} = \mathcal{C}(\mathcal{U})$ and each U_i is a convex open subset of $\mathcal{R}^d$, then $\mathcal{C}$ is a ***convex code***.

The ***minimum embedding dimension*** $d(\mathcal{C})$ of a convex code is the minimum dimension such that $\mathcal{C}$ admits a convex representation.

A code $\mathcal{C} = \mathcal{C}(\mathcal{U})$ has a ***local obstruction*** if there exists a nonempty intersection $U_\sigma = \bigcap_{i\in\sigma} U_i$ such that $U_\sigma \subseteq \bigcup_{i\in\tau} U_i$, but the nerve of the cover $\{U_i \cap U_\sigma\}_{i\in\tau}$ is *not* contractible.

Facts:

1. Convex codes have been observed in several brain areas. Orientation-selective neurons in the visual cortex [BeBaSo95] have convex receptive fields that reflect a neuron's preference for a particular angle. Hippocampal place cells [McEtal06], [OKDo71] are neurons that have spatial receptive fields, called *place fields*, that are typically convex.

2. If $\mathcal{C}$ has a local obstruction, then $\mathcal{C}$ is not a convex code.

3. All codes on $n \leq 2$ neurons are convex.

4. For $n \leq 3$ there are six non-convex codes (up to permutation-equivalence, and including the all-zeros word). They are $\{000, 010, 001, 110, 101\}$, $\{000, 010, 110, 101\}$, $\{000, 110, 101\}$, $\{000, 100, 010, 110, 101, 011\}$, $\{000, 100, 110, 101, 011\}$, $\{000, 110, 101, 011\}$.

Examples:

1. The code $\mathcal{C} = \mathcal{C}(\mathcal{U})$ shown in panel (B) of §19.6.2, Example 1 is convex by construction. Panel (A) shows a two-dimensional convex realization. The minimum embedding dimension for this code is $d(\mathcal{C}) = 2$.

2. Consider the code $\hat{\mathcal{C}} = \mathcal{C} \setminus \{0110\}$, where $\mathcal{C}$ is the code in Example 1. Code $\hat{\mathcal{C}}$ differs from $\mathcal{C}$ by only one codeword and has the same simplicial complex $\Delta(\hat{\mathcal{C}}) = \Delta(\mathcal{C})$. However, $\hat{\mathcal{C}}$ is not a convex code. It has a local obstruction because $U_2 \cap U_3 \subseteq U_1 \cup U_4$, yet the nerve of the cover of $U_\sigma = U_2 \cap U_3$ by $U_1 \cap U_\sigma$ and $U_4 \cap U_\sigma$ is disconnected, and thus not contractible.

3. The codes $\mathcal{C}_1 = \{111, 011, 001\}$ and $\mathcal{C}_2 = \{111, 101, 011, 110, 100, 010\}$ are both convex and have the same simplicial complex, but possess different embedding dimensions: $d(\mathcal{C}_1) = 1$, while $d(\mathcal{C}_2) = 2$.

Open Questions:

1. How do we determine, in general, whether or not a code is convex?

2. Are there other obstructions to convexity beyond local obstructions?

3. If a code is convex, what is the minimum embedding dimension?

19.6.5 FEEDFORWARD AND HYPERPLANE CODES

Hyperplane codes are an important class of combinatorial codes. These are codes that arise as an output of a one-layer feedforward neural network, and they are sometimes referred to as *feedforward codes*.

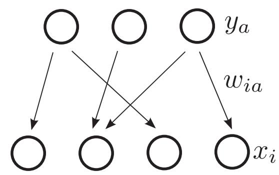
Definitions:

A ***hyperplane code*** is a convex code, where the underlying open convex cover $\mathcal{U} = \{U_i\}_{i=1}^n$ can be obtained as $U_i = X \cap H_i^+$, where $X \subseteq \mathcal{R}^m$ is an open convex set and the

$$H_i^+ = \{y \in \mathcal{R}^m \mid \sum_{a=1}^m w_{ia} y_a - \theta_i > 0\} \tag{1}$$

are open half-spaces.

A ***one-layer feedforward neural network*** is a network with input and output layers connected as shown in the following figure.



The network inputs nonnegative numbers $y_a \geq 0$ and outputs nonnegative numbers $x_i \geq 0$ according to the rule

$$x_i(y) = \phi\left(\sum_{a=1}^{m} w_{ia} y_a - \theta_i\right), \qquad i \in [n]. \tag{2}$$

Here $\theta_i \in \mathcal{R}$ are the neuronal thresholds, $w_{ia} \in \mathcal{R}$ are the effective strengths of the feedforward connections, and the ***transfer function*** $\phi \colon \mathcal{R} \to \mathcal{R}_{>0}$ satisfies the condition $\phi(t) = 0$ if $t \leq 0$ and $\phi(t) > 0$ if $t > 0$.

A ***feedforward code*** is a hyperplane code, where the underlying convex set X can be chosen to be the positive orthant $\mathcal{R}^m_+$. This class of codes arises as the output of a one-layer feedforward neural network (2), where positivity of each row of (2) corresponds to the halfspace H_i^+ in (1). Specifically the code of the network (2) is

$$\mathcal{C}(w, \theta) = \left\{\sigma \subseteq [n] \mid \exists y \in \mathcal{R}^m_+ \text{ such that } x_i(y) > 0 \Leftrightarrow i \in \sigma\right\}.$$

Facts:

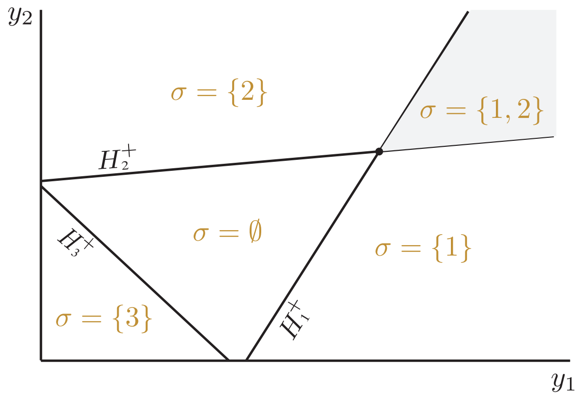
1. Hyperplane codes (and thus feedforward codes) are convex.

2. Not every convex code is a hyperplane code. Perhaps the smallest example is the code $\mathcal{C} = \{\varnothing, 2, 3, 4, 12, 13, 14, 123, 124\}$, which can be easily seen to possess a 2-dimensional convex realization. However, it can be proved to be not realizable as a hyperplane code.

3. *Theorem* [GiIt14]: For every simplicial complex K with n vertices, there exists a feedforward network (w, θ) described by (2) so that K is the simplicial complex of the appropriate feedforward code $K = \Delta(\mathcal{C}(w, \theta))$.

Example:

1. The following figure, adapted from [GiIt14], displays a feedforward code for a network with two neurons in the input layer, corresponding to the axes y_1 and y_2, and three neurons in the output layer, corresponding to the (oriented) hyperplanes H_1, H_2, and H_3. For each output neuron x_i, the inputs $y = (y_1, y_2)$ that yield $x_i(y) > 0$ lie in the positive halfspace H_i^+. The resulting code $\mathcal{C}$ consists of combinations of output neurons that can be simultaneously activated by at least one choice of nonnegative inputs.



$$\mathcal{C} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}\}$$
$$= \{000, 100, 010, 001, 110\}$$

REFERENCES

Printed Resources:

[BeBaSo95] R. Ben-Yishai, R. L. Bar-Or, and H. Sompolinsky, "Theory of orientation tuning in visual cortex", *Proceedings of the National Academy of Sciences USA* 92 (1995), 3844–3848.

[CuEtal15] C. Curto, E. Gross, J. Jeffries, K. Morrison, M. Omar, Z. Rosen, A. Shiu, and N. Youngs, "What makes a neural code convex?", preprint, Nov. 2015; available at `http://arxiv.org/abs/1508.00150`.

[CuIt08] C. Curto and V. Itskov, "Cell groups reveal structure of stimulus space", *PLoS Computational Biology* 4 (2008), e1000205.

[CuEtal13a] C. Curto, V. Itskov, K. Morrison, Z. Roth, and J. L. Walker, "Combinatorial neural codes from a mathematical coding theory perspective", *Neural Computation* 25 (2013), 1891–1925.

[CuEtal13b] C. Curto, V. Itskov, A. Veliz-Cuba, and N. Youngs, "The neural ring: an algebraic tool for analyzing the intrinsic structure of neural codes, *Bulletin of Mathematical Biology* 75 (2013), 1571–1611.

[CuYo15] C. Curto and N. Youngs, "Neural ring homomorphisms and maps between neural codes", preprint, Nov. 2015; available at `http://arxiv.org/abs/1511.00255`.

[GiIt14] C. Giusti and V. Itskov, "A no-go theorem for one-layer feedforward networks", *Neural Computation* 26 (2014), 2527–2540.

[Ka84] G. Kalai, "Characterization of f-vectors of families of convex sets in $\mathcal{R}^d$. I. Necessity of Eckhoff's conditions", *Israel Journal of Mathematics* 48 (1984), 175–195.

[Ka86] G. Kalai, Characterization of f-vectors of families of convex sets in $\mathcal{R}^d$. II. Sufficiency of Eckhoff's conditions", *Journal of Combinatorial Theory, Series A* 41 (1986), 167–188.

[Ma15] Y. I. Manin, "Neural codes and homotopy types: mathematical models of place field recognition", *Moscow Mathematical Journal* 15 (2015), 741–748.

[McEtal06] B. L. McNaughton, F. P. Battaglia, O. Jensen, E. I. Moser, and M. B. Moser, "Path integration and the neural basis of the 'cognitive map'", *Nature Reviews Neuroscience* 7 (2006), 663–678.

[MiSt05] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Springer, 2005.

[OKDo71] J. O'Keefe and J. Dostrovsky, "The hippocampus as a spatial map. Preliminary evidence from unit activity in the freely-moving rat", *Brain Research* 34 (1971), 171–175.

Web Resources:

`https://github.com/nebneuron/neural-ideal` (Software for computing with the neural ideal.