SUPPLEMENTARY MATERIALS: Sequential Attractors in Combinatorial Threshold-linear Networks*

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SM1. Appendix: Supplemental Materials.

SM1.1. Background on fixed points and simply-added splits.

Characterizations of fixed point supports. To exploit previous characterizations of fixed points in terms of their supports [SM2], we will restrict consideration to CTLNs that are **nondegenerate**, as defined below.

Definition SM1.1. *We say that a CTLN $W = W(G, \varepsilon, \delta)$ is* nondegenerate *if*

- $\det(I - W_\sigma) \neq 0$ *for each* $\sigma \subseteq [n]$*, and*
- *for each* $\sigma \subseteq [n]$ *and all* $i \in \sigma$*, the corresponding Cramer's determinant is nonzero:* $\det((I - W_\sigma)_i; \theta) \neq 0$.

Note that almost all CTLNs are nondegenerate, since having a zero determinant is a highly fine-tuned condition. The notation $\det(A_i; b)$ denotes the determinant obtained by replacing the i^{th} column of A with the vector b, as in Cramer's rule. In the case of a restricted matrix, $((A_\sigma)_i; b_\sigma)$ denotes the matrix obtained from A_σ by replacing the column corresponding to the index $i \in \sigma$ with b_σ (note that this is not typically the i^{th} column of A_σ).

When a CTLN is nondegenerate, there can be at most one fixed point per support. Specifically, if x^* is a fixed point with support σ, then for all $i \in \sigma$, we have $x^*_i = x^\sigma_i$ where

$$x^\sigma \stackrel{\text{def}}{=} \theta(I - W_\sigma)^{-1} 1_\sigma, \tag{SM1.1}$$

and for all $k \notin \sigma$, we have $x^*_k = 0$. (Note that 1_σ denotes the vector of all ones with length $|\sigma|$.) To check if a given subset $\sigma \subseteq [n]$ is the support of a fixed point of a CTLN $W = W(G, \varepsilon, \delta)$, one method is to compute the putative value of the fixed point via Equation (SM1.1) and see if it actually satisfies the TLN equations. Specifically, we see that σ is the support of a fixed point of W if and only if

(i) $x^\sigma_i > 0$ for all $i \in \sigma$ ("on"-neuron conditions), and
(ii) $\sum_{i \in \sigma} W_{ki} x^\sigma_i + \theta \leq 0$ for all $k \notin \sigma$ ("off"-neuron conditions).

(This is straightforward, but see [SM1] for more details.) Intuitively, σ is the support of a fixed point of the CTLN if the fixed point x^σ of the linear system restricted to σ has only positive entries, so that all the neurons in σ are "on" at the fixed point, and if the inputs to all the external nodes are sufficiently inhibitory (negative) to ensure that those external

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neurons remain "off". Since condition (i) above only depends on W_σ, a necessary condition for $\sigma \in \mathrm{FP}(G)$ is that $\sigma \in \mathrm{FP}(G|_\sigma)$, where $G|_\sigma$ refers to the subgraph of G obtained by restricting to the vertices of σ and the edges between them. A fixed point $\sigma \in \mathrm{FP}(G|_\sigma)$ *survives* the addition of other nodes $k \notin \sigma$ precisely when condition (ii) is satisfied.

Unfortunately, the "on" and "off"-neuron characterization of fixed point supports relies on actually solving for a fixed point using $(I - W_\sigma)^{-1}$, and thus is difficult to directly connect to the graph structure encoded in $W = W(G, \varepsilon, \delta)$. In [SM2], an alternative characterization was developed in terms of Cramer's determinants (which are directly related to the values of x_i^σ by Cramer's rule). Specifically, for any $\sigma \subseteq [n]$, we define s_i^σ to be the relevant Cramer's determinant:

$$s_i^\sigma \stackrel{\text{def}}{=} \det((I - W_{\sigma\cup\{i\}})_i; b_{\sigma\cup\{i\}}), \quad \text{for each} \quad i \in [n]. \tag{SM1.2}$$

In [SM2, Lemma 2], a formula for s_k^σ was proven that directly connects it to the relevant quantity in the "off"-neuron condition:

$$s_k^\sigma = \sum_{i\in\sigma} W_{ki} s_i^\sigma + \theta \det(I - W_\sigma) \text{ for any } k \in [n]. \tag{SM1.3}$$

Combining this with Cramer's rule, it was shown that $\mathrm{FP}(G)$ can be fully characterized in terms of the *signs* of the s_i^σ. It turns out these signs are also connected to the *index* of a fixed point. For each fixed point of a CTLN $W = W(G, \varepsilon, \delta)$, labeled by its support $\sigma \in \mathrm{FP}(G)$, we define the *index* as

$$\mathrm{idx}(\sigma) \stackrel{\text{def}}{=} \mathrm{sgn} \det(I - W_\sigma).$$

Since we assume our CTLNs are nondegenerate, $\det(I - W_\sigma) \neq 0$ and thus $\mathrm{idx}(\sigma) \in \{\pm 1\}$.

Theorem SM1.2 (sign conditions (Theorem 2 in [SM2])). *Let G be a graph on n neurons and $W = W(G, \varepsilon, \delta)$ be a CTLN with graph G. For any nonempty $\sigma \subseteq [n]$,*

$$\sigma \text{ is a permitted motif} \quad \Leftrightarrow \quad \mathrm{sgn}\, s_i^\sigma = \mathrm{sgn}\, s_j^\sigma \text{ for all } i, j \in \sigma.$$

When σ is permitted, $\mathrm{sgn}\, s_i^\sigma = \mathrm{sgn} \det(I - W_\sigma) = \mathrm{idx}(\sigma)$ for all $i \in \sigma$.
Furthermore,

$$\sigma \in \mathrm{FP}(G) \quad \Leftrightarrow \quad \mathrm{sgn}\, s_i^\sigma = \mathrm{sgn}\, s_j^\sigma = -\mathrm{sgn}\, s_k^\sigma \text{ for all } i, j \in \sigma,\ k \notin \sigma.$$

From this result, we immediately obtain the following corollary.

Corollary SM1.3 (Corollary 2 in [SM2]). *Let $\sigma \subseteq [n]$. The following are equivalent:*

1. $\sigma \in \mathrm{FP}(G)$
2. $\sigma \in \mathrm{FP}(G|_\tau)$ *for all* $\sigma \subseteq \tau \subseteq [n]$
3. $\sigma \in \mathrm{FP}(G|_\sigma)$ *and* $\sigma \in \mathrm{FP}(G|_{\sigma\cup k})$ *for all* $k \notin \sigma$
4. $\sigma \in \mathrm{FP}(G|_{\sigma\cup k})$ *for all* $k \notin \sigma$

This shows that for σ to support a fixed point of the full network, it must support a fixed point in its own subnetwork, as well as every other subnetwork in between. Moreover, by (3), it is possible to check survival just one external node k at a time. Note that survival of an added node k is fully determined by $\operatorname{sgn} s_k^\sigma$ by Theorem SM1.2. Moreover, since $s_k^\sigma = \sum_{i\in\sigma} W_{ki} s_i^\sigma + \theta \det(I - W_\sigma)$, we see that $\operatorname{sgn} s_k^\sigma$ only depends on the *outgoing edges* from σ to k (captured in W_{ki} values) as well as the edges within σ (reflected in s_i^σ and $\det(I - W_\sigma)$). Thus, only the outgoing edges from σ are relevant to its survival in a larger network.

Background on simply-added splits. It turns out that the s_i^σ are easy to compute when a graph has simply-added structure. Recall that in a simply-embedded partition, every node within a component receives identical incoming edges from the rest of the graph. This is a special case of the more general notion of a *simply-added split*.

Definition SM1.4 (simply-added split). *Let G be a graph on n nodes. For any nonempty ω, $\tau \subseteq [n]$ such that $\omega \cap \tau = \emptyset$, we say ω is* simply-added *onto τ if for each $j \in \omega$, either j is a* projector *onto τ, i.e., $j \to k$ for all $k \in \tau$, or j is a* nonprojector *onto τ, so $j \not\to k$ for all $k \in \tau$. In this case, we say that τ is* simply-embedded *in G, and we say that (ω, τ) is a* simply-added split *of the subgraph $G|_\sigma$, for $\sigma = \omega \cup \tau$.*

Note that when a graph has a simply-embedded partition $\{\tau_1|\cdots|\tau_N\}$, we have a simply-added split for every τ_i; specifically, $[n] \setminus \tau_i$ is simply-added onto τ_i, since by definition, τ_i is simply-embedded in G. In [SM2], it was shown that whenever a simply-added split exists, we can understand many of the s_i^σ values as scalings of s_i^τ from the smaller component subgraph $G|_\tau$.

Theorem SM1.5 (Theorem 3 in [SM2]). *Let G be a graph on n nodes, and let ω, $\tau \subseteq [n]$ be such that ω is simply-added to τ. For $\sigma \subseteq \omega \cup \tau$, define $\sigma_\omega \overset{\text{def}}{=} \sigma \cap \omega$ and $\sigma_\tau \overset{\text{def}}{=} \sigma \cap \tau$. Then*

$$s_i^\sigma = \frac{1}{\theta} s_i^{\sigma_\omega} s_i^{\sigma_\tau} = \alpha s_i^{\sigma_\tau} \quad \textit{for each } i \in \tau,$$

where $\alpha = \frac{1}{\theta} s_i^{\sigma_\omega}$ has the same value for every $i \in \tau$.

SM1.2. Proofs of Theorems 1.4 and other results on simply-embedded partitions. Theorem SM1.5 can immediately be leveraged for simply-embedded partitions to connect the s_j^σ values to the $s_j^{\sigma_i}$ values from the component subgraphs. This will be key to the proof of Theorem 1.4.

Lemma SM1.6. *Let G have a simply-embedded partition $\{\tau_1|\cdots|\tau_N\}$, and consider $\sigma \subseteq [n]$. Let $\sigma_i \overset{\text{def}}{=} \sigma \cap \tau_i$. Then for any $\sigma_i \neq \emptyset$,*

$$\operatorname{sgn} s_j^\sigma = \operatorname{sgn} s_k^\sigma \quad \Leftrightarrow \quad \operatorname{sgn} s_j^{\sigma_i} = \operatorname{sgn} s_k^{\sigma_i}, \quad \textit{for all } j,k \in \tau_i.$$

Proof. By definition of simply-embedded partition, G has a simply-added split where $[n]\setminus\tau_i$ is simply-added onto τ_i (and thus also onto σ_i). Thus by Theorem SM1.5, $s_j^\sigma = \alpha s_j^{\sigma_i}$, where $\alpha = \frac{1}{\theta} s_j^{\sigma\setminus\sigma_i}$ is identical for all $j \in \tau_i$. Hence, for all $j,k \in \tau_i$, we have that $\operatorname{sgn} s_j^\sigma = \operatorname{sgn} s_k^\sigma$ if and only if $\operatorname{sgn} \alpha s_j^{\sigma_i} = \operatorname{sgn} \alpha s_k^{\sigma_i}$ if and only if $\operatorname{sgn} s_j^{\sigma_i} = \operatorname{sgn} s_k^{\sigma_i}$.

Theorem 1.4 (reprinted below) now follows directly from Lemma SM1.6 together with the sign conditions characterization of fixed point supports (Theorem SM1.2).

Theorem 1.4 (FP(G) menu for simply-embedded partitions). *Let G have a simply-embedded partition $\{\tau_1|\cdots|\tau_N\}$. For any $\sigma \subseteq [n]$, let $\sigma_i \stackrel{\text{def}}{=} \sigma \cap \tau_i$. Then*

$$\sigma \in \text{FP}(G) \quad \Rightarrow \quad \sigma_i \in \text{FP}(G|_{\tau_i}) \cup \{\emptyset\} \quad \text{for all } i \in [N].$$

In other words, every fixed point support of G is a union of component fixed point supports σ_i, at most one per component.

Proof. For $\sigma \in \text{FP}(G)$, we have

$$\operatorname{sgn} s_j^\sigma = \operatorname{sgn} s_k^\sigma = -\operatorname{sgn} s_l^\sigma$$

for any $j, k \in \sigma_i$ and $l \in \tau_i \setminus \sigma_i$, by Theorem SM1.2 (sign conditions). Then by Lemma SM1.6, we see that whenever $\sigma_i \neq \emptyset$,

$$\operatorname{sgn} s_j^{\sigma_i} = \operatorname{sgn} s_k^{\sigma_i} = -\operatorname{sgn}_l^{\sigma_i},$$

and so σ_i satisfies the sign conditions in $G|_{\tau_i}$. Thus $\sigma_i \in \text{FP}(G|_{\tau_i})$ for every nonempty σ_i. ■

Next we prove that whenever a graph G has a simply-embedded partition and there is a locally removable node (i.e. a node whose removal does not affect its component $\text{FP}(G|_{\tau_i})$), then that node is also globally removable with no impact on $\text{FP}(G)$ (Theorem 3.3 reprinted below for convenience).

Theorem 3.3 (removable nodes). *Let G have a simply-embedded partition $\{\tau_1|\cdots|\tau_N\}$. Suppose there exists a node $j \in \tau_i$ such that $\text{FP}(G|_{\tau_i}) = \text{FP}(G|_{\tau_i\setminus\{j\}})$. Then $\text{FP}(G) = \text{FP}(G|_{[n]\setminus\{j\}})$.*

Proof. To see that $\text{FP}(G) \subseteq \text{FP}(G|_{[n]\setminus\{j\}})$, notice that for all $\sigma \in \text{FP}(G)$, we have $\sigma \subseteq [n] \setminus \{j\}$ by Theorem 1.4. Then by Corollary SM1.3(2), we must have $\sigma \in \text{FP}(G|_{[n]\setminus\{j\}})$, and so $\text{FP}(G) \subseteq \text{FP}(G|_{[n]\setminus\{j\}})$.

For the reverse containment, we will show that every fixed point in $\text{FP}(G|_{[n]\setminus\{j\}})$ survives the addition of node j by appealing to Theorem SM1.2 (sign conditions). There are two cases to consider: $\sigma_i = \emptyset$ and $\sigma_i \neq \emptyset$, where $j \in \tau_i$ and $\sigma_i \stackrel{\text{def}}{=} \sigma \cap \tau_i$.

Case 1: $\sigma_i = \emptyset$. Since j is not contained in the support of any fixed point of $G|_{\tau_i}$, there must be at least one other node k in τ_i, since $\text{FP}(G|_{\tau_i})$ cannot be empty. Since G is a simply-embedded partition, we have that $[n] \setminus \tau_i$ is simply-embedded onto τ_i meaning that every node in τ_i receives identical inputs from the rest of the graph. Recall from Equation (SM1.3), that $s_j^\sigma = \sum_{\ell\in\sigma} W_{j\ell} s_\ell + \theta \det(I - W_\sigma)$. Then since $\sigma \subseteq [n] \setminus \tau_i$, we have that j and k receive identical inputs from σ, so $W_{j\ell} = W_{k\ell}$ for all $\ell \in \sigma$, and thus $s_j^\sigma = s_k^\sigma$. Since $\sigma \in \text{FP}(G|_{[n]\setminus\{j\}})$, we have $\operatorname{sgn} s_k^\sigma = -\operatorname{sgn} s_\ell^\sigma$ for all $\ell \in \sigma$ by Theorem SM1.2 (sign conditions). Thus, we also have $\operatorname{sgn} s_j^\sigma = -\operatorname{sgn} s_\ell^\sigma$ and σ survives the addition of node j, so $\sigma \in \text{FP}(G)$.

Case 2: $\sigma_i \neq \emptyset$. First observe that $G|_{[n]\setminus\{j\}}$ has the same simply-embedded partition structure as G, but with $\tau_i \setminus \{j\}$ rather than τ_i. Thus $\sigma \in \text{FP}(G|_{[n]\setminus\{j\}})$ implies that $\sigma_i \in \text{FP}(G|_{\tau_i\setminus\{j\}})$ by Theorem 1.4 (menu). By hypothesis, $\text{FP}(G|_{\tau_i\setminus\{j\}}) = \text{FP}(G|_{\tau_i})$, and so $\sigma_i \in \text{FP}(G|_{\tau_i})$. Then by Theorem SM1.2 (sign conditions), since $j \notin \sigma_i$, we have $\operatorname{sgn} s_j^{\sigma_i} = -\operatorname{sgn} s_\ell^{\sigma_i}$ for all $\ell \in \sigma_i$. And by Lemma SM1.6, this ensures $\operatorname{sgn} s_j^\sigma = -\operatorname{sgn} s_\ell^\sigma$ for all $\ell \in \sigma_i$. Since $\sigma \in \text{FP}(G|_{[n]\setminus\{j\}})$, we have that $\operatorname{sgn} s_\ell^\sigma$ is identical for all $\ell \in \sigma$, not just $\ell \in \sigma_i$, and so $\operatorname{sgn} s_j^\sigma = -\operatorname{sgn} s_\ell^\sigma$ for all $\ell \in \sigma$. Thus by Theorem SM1.2 (sign conditions), σ survives the addition of node j, so $\sigma \in \mathrm{FP}(G)$. ∎

Corollary 3.4. *Let G have a simply-embedded partition $\{\tau_1|\cdots|\tau_N\}$ and suppose there exists $j \in \tau_i$ such that $\mathrm{FP}(G|_{\tau_i}) = \mathrm{FP}(G|_{\tau_i \setminus \{j\}})$. Let G' be any graph that can be obtained from G by deleting or adding all the outgoing edges from j to any component τ_k with $k \neq i$. Then $\mathrm{FP}(G') = \mathrm{FP}(G)$.*

Proof. Observe that by deleting all the outgoing edges from j to a component τ_k, node j has simply changed from a projector onto τ_k to a nonprojector. Alternatively, by adding all the outgoing edges to τ_k, node j switches from being a nonprojector onto τ_k to being a projector. In either case, j is still simply-added onto τ_k, and so G' has the same simply-embedded partition $\{\tau_1|\cdots|\tau_N\}$ as G had. Additionally, since no edges within τ_i have been altered, we have that $\mathrm{FP}(G'|_{\tau_i}) = \mathrm{FP}(G|_{\tau_i}) = \mathrm{FP}(G|_{\tau_i\setminus\{j\}}) = \mathrm{FP}(G'|_{\tau_i\setminus\{j\}})$. Thus both G and G' satisfy the hypotheses of Theorem 3.3. Moreover, $G|_{[n]\setminus\{j\}} = G'|_{[n]\setminus\{j\}}$ since the only differences between G and G' were in edges involving node j, which has been removed. Thus, by Theorem 3.3, $\mathrm{FP}(G) = \mathrm{FP}(G|_{[n]\setminus\{j\}}) = \mathrm{FP}(G')$. ∎

SM1.3. Background on bidirectional simply-added splits. In order to prove the properties of $\mathrm{FP}(G)$ for simple linear chains and strongly simply-embedded partitions, we first need to review some background from [SM2] on *bidirectional simply-added splits*. These are partitions into two components in which each component is simply-added onto the other component (so the simply-added property is bidirectional).

Definition SM1.7 (bidirectional simply-added split). *Let G be a graph on n nodes. For any nonempty ω, $\tau \subseteq [n]$ such that $[n] = \omega \cup \tau$ and $\omega \cap \tau = \emptyset$, we say that G has a* bidirectional simply-added split (ω, τ) *if ω is simply-added onto τ and τ is simply-added onto ω. In other words, for all $j \in \omega$, either $j \to k$ for all $k \in \tau$ or $j \not\to k$ for all $k \in \tau$,* and *for all $k \in \tau$, either $k \to j$ for all $j \in \omega$ or $k \not\to j$ for all $j \in \omega$.*

Note that a simply-embedded partition consisting of just two components $\{\tau_1 \mid \tau_2\}$ is a bidirectional simply-added split. But with larger simply-embedded partitions, $\{\tau_1|\cdots|\tau_N\}$, it is not generally true that $(\tau_i, [n] \setminus \tau_i)$ is a bidirectional simply-added split. However, *strongly simply-embedded partitions* will always satisfy that $(\tau_i, [n] \setminus \tau_i)$ is a bidirectional simply-added

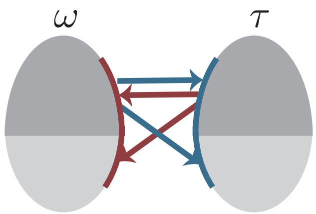

Figure SM1. ***Bidirectional simply-added split.*** *In this graph ω is simply-added to τ and vice versa. Thus ω is composed of two classes of nodes: projectors onto τ (top dark gray region) and nonprojectors onto τ (bottom light gray region). Similarly, τ can be decomposed into projectors and nonprojectors onto ω. The thick colored arrows indicate that every node of a given region sends an edge to every node in the other region. The edges within ω and τ can be arbitrary.*

split. This is because in a strongly simply-embedded partition, any $j \in \tau_i$ treats all the other components identically, so it is either a projector or a non-projector onto all of $[n] \setminus \tau_i$.

In [SM2], it was shown that $\text{FP}(G)$ is fully determined by the fixed points of the component subgraphs $G|_\omega$ and $G|_\tau$ when (ω, τ) is a bidirectional simply-added split. To make this characterization precise, we first need some notation. For any $\omega \subseteq [n]$, let S_ω denote the fixed point supports of $G|_\omega$ that survive to be fixed points of G, and let D_ω denote the non-surviving (dying) fixed points:

$$S_\omega \stackrel{\text{def}}{=} \text{FP}(G|_\omega) \cap \text{FP}(G), \quad \text{and} \quad D_\omega \stackrel{\text{def}}{=} \text{FP}(G|_\omega) \setminus S_\omega.$$

Theorem SM1.8 (Theorem 14 in [SM2]). *Let G be a graph with bidirectional simply-added split $[n] = \omega \cup \tau$. For any nonempty $\sigma \subseteq [n]$, let $\sigma = \sigma_\omega \cup \sigma_\tau$ where $\sigma_\omega \stackrel{\text{def}}{=} \sigma \cap \omega$ and $\sigma_\tau \stackrel{\text{def}}{=} \sigma \cap \tau$. Then $\sigma \in \text{FP}(G)$ if and only if one of the following holds:*

(i) $\sigma_\tau \in S_\tau \cup \{\emptyset\}$ *and* $\sigma_\omega \in S_\omega \cup \{\emptyset\}$, *or*

(ii) $\sigma_\tau \in D_\tau$ *and* $\sigma_\omega \in D_\omega$.

In other words, $\sigma \in \text{FP}(G)$ if and only if σ is either a union of surviving fixed points σ_i, at most one from ω and at most one from τ, or it is a union of dying fixed points, exactly one from ω and one from τ.

We will see that both simple linear chains and strongly simply-embedded partitions have bidirectional simply-added splits within them, and so Theorem SM1.8 will be key to the proofs characterizing their $\text{FP}(G)$. First, though, we take a brief detour to explore the special case of bidirectional simply-added splits with singletons in a component, in order to see some special internal structure of $\text{FP}(G)$ in these cases.

SM1.4. Internal structure of $\text{FP}(G)$ with singletons. A special case of a bidirectional simply-added split occurs whenever a graph contains a node that is projector/nonprojector onto the rest of the graph. Specifically, since any subset is always simply-added onto a single node j trivially, we see that we have a bidirectional simply-added split $(\{j\}, [n] \setminus \{j\})$ whenever j is either a projector or a nonprojector onto the rest of the graph. Recall that if j is a nonprojector onto $[n] \setminus \{j\}$, then j has no outgoing edges in G, and so it is a *sink*. Moreover, we have seen that sinks are the only single nodes that can support fixed points since a singleton $\{j\}$ is trivially uniform in-degree 0, and thus only survives when it has no outgoing edges, by Rule 1. Combining this observation with the bidirectional simply-added split for a sink, we see there is certain internal structure that must be present in $\text{FP}(G)$ whenever it contains any singleton sets.

Proposition SM1.9. *Let G be a graph such that there is some singleton $\{j\} \in \text{FP}(G)$. Then for any $\sigma \in \text{FP}(G)$ (with $\sigma \neq \{j\}$),*

(1) *If $j \notin \sigma$, then $\sigma \cup \{j\} \in \text{FP}(G)$; i.e., $\text{FP}(G)$ is closed under unions with singletons.*

(2) *If $j \in \sigma$, then $\sigma \setminus \{j\} \in \text{FP}(G)$; i.e., $\text{FP}(G)$ is closed under set differences with singletons.*

Proof. First notice that since $\{j\} \in \text{FP}(G)$, j is a sink in G by Rule 1 (since a singleton is trivially uniform in-degree 0, and thus survives exactly when it has no outgoing edges), and therefore $(\{j\}, [n] \setminus \{j\})$ is a bidirectional simply-added split.

To prove (1), suppose $j \notin \sigma$. Since $(\{j\}, [n] \setminus \{j\})$ is a bidirectional simply-added split, Theorem SM1.8 guarantees that $\sigma \cup \{j\} \in \text{FP}(G)$ if and only if $\{j\}, \sigma$ both survive or both die. By assumption, both sets are in $\text{FP}(G)$, so both survive. Thus, $\sigma \cup \{j\} \in \text{FP}(G)$.

To prove (2), suppose $j \in \sigma$. By Theorem SM1.8, $\sigma \in \text{FP}(G)$ if and only if $\{j\}, \sigma \setminus \{j\}$ both survive or both die. By assumption, $\{j\} \in \text{FP}(G)$, and so $\sigma \setminus \{j\} \in \text{FP}(G)$ as well. ∎

Corollary SM1.10. *Let G be a graph such that $\text{FP}(G)$ contains singleton sets $\{j_1\}, \{j_2\}, \ldots, \{j_\ell\}$, and let $\mathcal{S} = \{j_1, \ldots, j_\ell\}$ be the set of singletons. Then for any $\sigma \in \text{FP}(G)$ and any $\omega \subseteq \mathcal{S}$*

$$\sigma \cup \omega \in \text{FP}(G).$$

Moreover, let $\tau = [n] \setminus \mathcal{S}$. Then $\text{FP}(G)$ has the direct product structure:

$$\text{FP}(G) \cup \{\emptyset\} \cong (\{\sigma \in \text{FP}(G|_\tau) \mid \sigma \in \text{FP}(G)\} \cup \{\emptyset\}) \times \mathcal{P}(\mathcal{S}),$$

where $\mathcal{P}(\mathcal{S})$ denotes the power set of $\mathcal{S}$. In other words, every fixed point support in $\text{FP}(G)$ has the form $\sigma \cup \omega$ where $\sigma \in \text{FP}(G|_\tau) \cup \{\emptyset\}$ and $\omega \subseteq \mathcal{S}$.

Proof. The first statement follows by iterating Proposition SM1.9(1) $|\omega|$ times for each of the added singletons in ω. To prove the second statement, we will show that every $\nu \in \text{FP}(G)$ is the union of a surviving fixed point $\sigma \subseteq \tau$ (or the empty set) with a subset of $\mathcal{S}$ (including empty set); moreover, every such union yields a fixed point (other than $\emptyset \cup \emptyset$). The direct product structure of $\text{FP}(G)$ immediately follows from this decomposition of the fixed point supports. By the first result, we see that every such union is contained in $\text{FP}(G)$. Thus, all that remains to show is that every element of $\text{FP}(G)$ is such a union. Let $\nu \in \text{FP}(G)$ and let $\sigma = \nu \cap \tau$ and $\omega = \nu \cap \mathcal{S}$, so that $\nu = \sigma \cup \omega$. If σ or ω are empty, then we're done, so suppose both are nonempty. Then we can iteratively apply Proposition SM1.9(2) $|\omega|$ times to see that $\sigma \in \text{FP}(G)$. Thus, every fixed point support arises as a union of some $\sigma \subseteq \tau$ with an arbitrary subset of $\mathcal{S}$, where $\sigma \in \text{FP}(G) \cup \{\emptyset\}$ (and for every $\sigma \in \text{FP}(G)$, we have $\sigma \in \text{FP}(G|_\tau)$ as well by Corollary SM1.3(2)). ∎

SM1.5. Simple linear chain proofs. In this section, we prove Theorem 3.6 showing that $\text{FP}(G)$ for a simple linear chain is closed under unions of component fixed points σ_i that survive in $G|_{\tau_i \cup \tau_{i+1}}$. The proof relies on the existence of a bidirectional simply-added split within a simple linear chain between the first $N-1$ components of the chain and τ_N.

Another key to the proof is the fact that if $\sigma_i \in \text{FP}(G|_{\tau_i \cup \tau_{i+1}})$, then it turns out that $\sigma_i \in \text{FP}(G)$; in other words, survival of the addition of the next component is sufficient to guarantee survival in the full network. This occurs because σ_i has no outgoing edges to any nodes outside of $\tau_i \cup \tau_{i+1}$. Lemma SM1.11 shows that whenever a permitted motif has no outgoing edges to a node k, then it is guaranteed to survive the addition of node k.

Lemma SM1.11. *Let G be a graph on n nodes, let $\sigma \subseteq [n]$ be nonempty, and $k \in [n] \setminus \sigma$. If $i \not\to k$ for all $i \in \sigma$, then*

$$\sigma \in \text{FP}(G|_{\sigma \cup \{k\}}) \quad \Leftrightarrow \quad \sigma \in \text{FP}(G|_\sigma).$$

In other words, if σ has no outgoing edges to node k then σ is guaranteed to survive the addition of node k whenever σ is a permitted motif.

Proof. For any $j \in \sigma$, we have that j inside-out dominates k. Thus by Rule 2c, $\sigma \in \mathrm{FP}(G|_{\sigma\cup\{k\}})$ if and only if $\sigma \in \mathrm{FP}(G|_\sigma)$. ∎

The proof of Lemma SM1.11 illustrated how inside-out graphical domination can be used to guarantee survival of a permitted motif. The presence of such a graphical domination relationship is a sufficient condition to guarantee survival, but unfortunately it is not a necessary condition, so the absence of such a relationship does not guarantee that a permitted motif does *not* survive. It turns out though, that graphical domination is a special case of *general domination*, and the presence/absence of a general domination relationship does precisely characterize survival of a fixed point support. To complete the proof of Theorem 3.6, we must appeal to general domination, and so we briefly review that concept here and the complete characterization of fixed point supports that it provides. (For a more detailed discussion of general domination, see section 6 of [SM2]).

Recall that Theorem SM1.2 (sign conditions) gives a complete characterization of when a subset σ supports a fixed point in terms of the *signs* of the Cramer's determinants s_i^σ. For general domination, these Cramer's determinants again play a key role, but in this case it will be the *magnitudes* of s_i^σ that are relevant, irrespective of their signs. Specifically, for any $j \in [n]$, we define the relevant domination quantity:

$$w_j^\sigma = \sum_{i\in\sigma} \widetilde{W}_{ji}|s_i^\sigma|,$$

where $\widetilde{W} = -I + W$, so that $\widetilde{W}_{ji} = W_{ji}$ if $j \neq i$ and $\widetilde{W}_{ji} = -1$ if $j = i$.

We say that k *dominates* j *with respect to* σ, if $w_k^\sigma > w_j^\sigma$. It turns out that $\sigma \in \mathrm{FP}(G)$ precisely when these domination quantities are perfectly balanced within σ, so that σ is *domination-free*, and when every external node $k \notin \sigma$ is inside-out dominated by nodes inside σ:

Theorem SM1.12 (general domination ([Theorem 15 in [SM2])). *Let G be a graph on n neurons and $W = W(G, \varepsilon, \delta)$ be a CTLN with graph G, and consider $\sigma \subseteq [n]$. Let $\widetilde{W} = -I+W$ and w_j^σ be as above. Then*

$$\sigma \in \mathrm{FP}(G|_\sigma) \quad \Leftrightarrow \quad w_i^\sigma = w_j^\sigma \text{ for all } i, j \in \sigma.$$

If $\sigma \in \mathrm{FP}(G|_\sigma)$, then $\sigma \in \mathrm{FP}(G)$ if and only if for each $k \notin \sigma$, there exists $j \in \sigma$ such that $w_j^\sigma > w_k^\sigma$, i.e. such that j inside-out dominates k.

It turns out that the simply-embedded partition structure of the simple linear chain with the added restriction that τ_i does not send edges to any τ_k other than τ_{i+1} gives significant structure to the values of s_i^σ and thus to the domination quantities w_j^σ. This structure is the key to the proof of Theorem 3.6.

Theorem 3.6 (simple linear chains). *Let G be a simple linear chain with components $\tau_1, \ldots, \tau_N$.*

(i) *If $\sigma \in \mathrm{FP}(G)$, then $\sigma_i \in \mathrm{FP}(G|_{\tau_i}) \cup \{\emptyset\}$ for all $i \in [N]$, where $\sigma_i = \sigma \cap \tau_i$.*

(ii) *Consider a collection* $\{\sigma_i\}_{i\in[N]}$ *of* $\sigma_i \in \text{FP}(G|_{\tau_i}) \cup \{\emptyset\}$. *If additionally* $\sigma_i \in \text{FP}(G|_{\tau_i\cup\tau_{i+1}}) \cup \{\emptyset\}$ *for all* $i \in [N]$*, then*

$$\bigcup_{i\in[N]} \sigma_i \in \text{FP}(G).$$

In other words, $\text{FP}(G)$ *is closed under unions of component fixed point supports that survive in* $G|_{\tau_i\cup\tau_{i+1}}$.

Proof. (i) follows directly from Theorem 1.4 by noting that the simple linear chain structure endows G with a simply-embedded partition: for every τ_i, the nodes in τ_{i-1} are each either a projector or nonprojector onto τ_i, while all nodes outside of τ_{i-1} are all nonprojectors onto τ_i.

To prove (ii), consider $\{\sigma_i\}_{i\in[N]}$ where $\sigma_i \in \text{FP}(G|_{\tau_i\cup\tau_{i+1}}) \cup \{\emptyset\}$ for all $i \in [N]$. Notice that by Lemma SM1.11, the fact that $\sigma_i \in \text{FP}(G|_{\tau_i\cup\tau_{i+1}})$ implies that $\sigma_i \in \text{FP}(G)$ since σ_i has no outgoing edges to any external node k outside of $\tau_i \cup \tau_{i+1}$. Thus, we may assume $\sigma_i \in \text{FP}(G) \cup \{\emptyset\}$ for all $i \in [N]$. We will prove that this guarantees that $\cup_{i\in[N]}\sigma_i \in \text{FP}(G)$ by induction on the number N of components of the simple linear chain.

For $N = 1$, the result is trivially true. For $N = 2$, observe that the simple linear chain on $\{\tau_1 \mid \tau_2\}$ actually has the structure of a bidirectional simply-embedded split (τ_1, τ_2), and thus Theorem SM1.8 gives the complete structure of $\text{FP}(G)$ in terms of the surviving fixed points of the component subgraphs S_{τ_i} and the dying fixed points D_{τ_i}. The sets of interest here, $\sigma_i \subseteq \tau_i$ with $\sigma_i \in \text{FP}(G)$, are precisely the elements of S_{τ_i}. Theorem SM1.8(1) then guarantees that $\sigma_1 \cup \sigma_2 \in \text{FP}(G)$ whenever $\sigma_i \in \text{FP}(G)$, and so the result holds when $N = 2$.

Now, suppose the result holds for any simple linear chain with $N - 1$ components. For ease of notation, denote $\sigma_{1\cdots N-1} \stackrel{\text{def}}{=} \sigma_1 \cup \cdots \cup \sigma_{N-1}$ and let $\sigma \stackrel{\text{def}}{=} \cup_{i\in[N]}\sigma_i$. We will show the result holds for any simple linear chain G with N components.

Observe that if $\sigma_N = \emptyset$, we have $\sigma = \sigma_{1\cdots N-1} \in \text{FP}(G|_{\tau_{1\cdots N-1}})$ by the inductive hypothesis, and we need only show that this implies that $\sigma_{1\cdots N-1} \in \text{FP}(G)$. On the other hand, if $\sigma_N \neq \emptyset$, then $\sigma = \sigma_{1\cdots N-1} \cup \sigma_N$, where $\sigma_N \in \text{FP}(G)$ by Lemma SM1.11, since $\sigma_N \in \text{FP}(G|_{\tau_N})$ and σ_N has no outgoing edges to any external nodes outside of τ_N. Notice that the simple linear chain structure of G ensures that $(\tau_{1\cdots N-1}, \tau_N)$ is a bidirectional simply-embedded split. Thus by Theorem SM1.8, since σ_N is a surviving fixed point support, $\sigma_{1\cdots N-1} \cup \sigma_N \in \text{FP}(G)$ if and only if $\sigma_{1\cdots N-1} \in \text{FP}(G)$. Therefore for any $\{\sigma_i\}_{i\in[N]}$, it suffices to show that $\sigma_{1\cdots N-1} \in \text{FP}(G)$, and the result will follow.

Notice that by the inductive hypothesis, $\sigma_{1\cdots N-1} \in \text{FP}(G|_{\tau_{1\cdots N-1}})$, and thus to show $\sigma_{1\cdots N-1} \in \text{FP}(G)$, we need only show that $\sigma_{1\cdots N-1}$ survives the addition of the nodes in τ_N. There are two cases to consider here based on whether $\sigma_{1\cdots N-1}$ intersects τ_{N-1} or not. Observe that if $\sigma_{1\cdots N-1} \cap \tau_{N-1} = \emptyset$, then $\sigma_{1\cdots N-1}$ has no outgoing edges to τ_N since only nodes in τ_{N-1} can send edges forward to τ_N by the linear chain structure. In this case, we have $i \not\to k$ for all $i \in \sigma_{1\cdots N-1}$ and all $k \in \tau_N$, and so Lemma SM1.11 guarantees that $\sigma_{1\cdots N-1} \in \text{FP}(G)$ since we already had $\sigma_{1\cdots N-1} \in \text{FP}(G|_{\tau_{1\cdots N-1}})$.

For the other case where $\sigma_{1\cdots N-1} \cap \tau_{N-1} \neq \emptyset$, we will prove $\sigma_{1\cdots N-1} \in \text{FP}(G)$ by appealing to Theorem SM1.12 (general domination) and demonstrating that each $k \in \tau_N$ is *inside-out* *dominated* by some node $j \in \sigma_{1\cdots N-1}$. First notice that $\sigma_{1\cdots N-1} = \sigma_{1\cdots N-2} \cup \sigma_{N-1}$ and by the simple linear chain structure of G, we have that $\tau_{1\cdots N-2}$ is simply-embedded onto τ_{N-1}. Thus by Theorem SM1.5,

$$s_i^{\sigma_{1\cdots N-1}} = \frac{1}{\theta} s_i^{\sigma_{1\cdots N-2}} s_i^{\sigma_{N-1}} = \alpha s_i^{\sigma_{N-1}} \text{ for all } i \in \sigma_{N-1}, \tag{SM1.4}$$

where $\alpha = \frac{1}{\theta} s_i^{\sigma_{1\cdots N-2}}$ has the same value for every $i \in \sigma_{N-1}$. Using this, we can now compute the domination quantities $w_j^{\sigma_{1\cdots N-1}}$ and $w_k^{\sigma_{1\cdots N-1}}$ for $j \in \sigma_{N-1}$ and $k \in \tau_N$. For $j \in \sigma_{N-1}$, we have:

$$\begin{aligned}
w_j^{\sigma_{1\cdots N-1}} &\overset{\text{def}}{=} \sum_{i \in \sigma_{1\cdots N-1}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| + \sum_{i \in \sigma_{N-1}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| + \sum_{i \in \sigma_{N-1}} \widetilde{W}_{ji} |\alpha s_i^{\sigma_{N-1}}| \quad \text{by (SM1.4)} \\
&= \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| + |\alpha| \sum_{i \in \sigma_{N-1}} \widetilde{W}_{ji} |s_i^{\sigma_{N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| + |\alpha| w_j^{\sigma_{N-1}}
\end{aligned}$$

On the other hand, for $k \in \tau_N$ we have the following formula for $w_k^{\sigma_{1\cdots N-1}}$, where we use the fact that $\widetilde{W}_{ki} = -1 - \delta$ for all $i \in \sigma_{1\cdots N-2}$ since there are no edges from nodes in $\tau_{1\cdots N-2}$ to τ_N:

$$\begin{aligned}
w_k^{\sigma_{1\cdots N-1}} &\overset{\text{def}}{=} \sum_{i \in \sigma_{1\cdots N-1}} \widetilde{W}_{ki} |s_i^{\sigma_{1\cdots N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ki} |s_i^{\sigma_{1\cdots N-1}}| + \sum_{i \in \sigma_{N-1}} \widetilde{W}_{ki} |s_i^{\sigma_{1\cdots N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} (-1-\delta) |s_i^{\sigma_{1\cdots N-1}}| + \sum_{i \in \sigma_{N-1}} \widetilde{W}_{ki} |\alpha s_i^{\sigma_{N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} (-1-\delta) |s_i^{\sigma_{1\cdots N-1}}| + |\alpha| \sum_{i \in \sigma_{N-1}} \widetilde{W}_{ki} |s_i^{\sigma_{N-1}}| \\
&= \sum_{i \in \sigma_{1\cdots N-2}} (-1-\delta) |s_i^{\sigma_{1\cdots N-1}}| + |\alpha| w_k^{\sigma_{N-1}}.
\end{aligned}$$

Moreover, since $\sigma_{N-1} \in \mathrm{FP}(G)$, we have that $j \in \sigma_{N-1}$ must inside-out dominate the external node k, so $w_j^{\sigma_{N-1}} > w_k^{\sigma_{N-1}}$. Combining this with the fact that $\widetilde{W}_{ji} \geq -1 - \delta$, we see that

$$\begin{aligned}
w_k^{\sigma_{1\cdots N-1}} &\leq \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| + |\alpha| w_k^{\sigma_{N-1}} \\
&< \sum_{i \in \sigma_{1\cdots N-2}} \widetilde{W}_{ji} |s_i^{\sigma_{1\cdots N-1}}| + |\alpha| w_j^{\sigma_{N-1}} = w_j^{\sigma_{1\cdots N-1}}
\end{aligned}$$

Thus $w_j^{\sigma_{1\cdots N-1}} > w_k^{\sigma_{1\cdots N-1}}$ and so j inside-out dominates k for all $k \in \tau_N$. Thus by Theorem SM1.12, $\sigma_{1\cdots N-1} \in \mathrm{FP}(G)$, and so $\cup_{i\in[N]}\sigma_i = \sigma_{1\cdots N-1} \cup \sigma_N \in \mathrm{FP}(G)$ as desired. ∎

SM1.6. Proofs for strongly simply-embedded partitions. In this section we prove Theorem 3.8, characterizing $\mathrm{FP}(G)$ for strongly simply-embedded partitions. First, we prove Lemma SM1.13 which shows that the strongly simply-embedded structure guarantees a complete factorization of the s_j^σ values in terms of the $s_j^{\sigma_i}$ of the component fixed point supports. Moreover, the $s_j^{\sigma_i}$ values are fully determined by whether σ_i is a surviving or a dying fixed point of $G|_{\tau_i}$. Recall that we denote the sets of surviving and dying fixed points as:

$$S_{\tau_i} \stackrel{\text{def}}{=} \mathrm{FP}(G|_{\tau_i}) \cap \mathrm{FP}(G) \quad \text{and} \quad D_{\tau_i} \stackrel{\text{def}}{=} \mathrm{FP}(G|_{\tau_i}) \setminus S_{\tau_i}.$$

Lemma SM1.13. *Let G be a graph on n nodes with a strongly simply-embedded partition $\{\tau_1|\ldots|\tau_N\}$. For any $\sigma \subseteq [n]$, denote $\sigma_i \stackrel{\text{def}}{=} \sigma \cap \tau_i$, and $\sigma_{i_1\cdots k} \stackrel{\text{def}}{=} \sigma_{i_1} \cup \cdots \cup \sigma_{i_k}$ and let $I = \{i \in [N] \mid \sigma_i \neq \emptyset\}$. Then for every $j \in [n]$,*

$$s_j^\sigma = \frac{1}{\theta^{|I|-1}} \prod_{i\in I} s_j^{\sigma_i},$$

where $s_j^{\sigma_i}$ has the same value for every $j \in [n] \setminus \tau_i$.
Moreoever, for any $\sigma_i \in \mathrm{FP}(G|_{\tau_i})$ and $j \in \tau_i$:

$$\operatorname{sgn} s_j^{\sigma_i} = \begin{cases} \mathrm{idx}(\sigma_i) & \text{if } j \in \sigma_i \\ -\mathrm{idx}(\sigma_i) & \text{if } j \in \tau_i \setminus \sigma_i \end{cases}$$

while for any $k \notin \tau_i$,

$$\operatorname{sgn} s_k^{\sigma_i} = \begin{cases} -\mathrm{idx}(\sigma_i) & \text{if } \sigma_i \in S_{\tau_i} \\ \mathrm{idx}(\sigma_i) & \text{if } \sigma_i \in D_{\tau_i} \end{cases}$$

Proof. Since $\{\tau_1|\ldots|\tau_N\}$ is a strongly simply-embedded partition of G, we have $[n] \setminus \tau_1$ simply-added onto τ_1, and so

$$s_j^\sigma = \frac{1}{\theta} s_j^{\sigma_{2\ldots N}} s_j^{\sigma_1} \text{ for all } j \in \tau_1$$

by Theorem SM1.5. On the other hand, since τ_1 is also simply-added onto $[n] \setminus \tau_1$, we also have

$$s_j^\sigma = \frac{1}{\theta} s_j^{\sigma_1} s_j^{\sigma_{2\ldots N}} \text{ for all } j \in [n] \setminus \tau_1.$$

Therefore, the above factorization holds for all $j \in [n]$. Similarly, since $[n] \setminus \tau_2$ is simply-added to τ_2 and vice versa,

$$s_j^{\sigma_{2\ldots N}} = \frac{1}{\theta} s_j^{\sigma_2} s_j^{\sigma_{3\ldots N}} \text{ for all } j\in[n]$$

by Theorem SM1.5, and so $s_j^\sigma = \frac{1}{\theta^2} s_j^{\sigma_1} s_j^{\sigma_2} s_j^{\sigma_{3\ldots N}}$. Continuing in this fashion, we see that for any $j \in [n]$,

$$s_j^\sigma = \frac{1}{\theta^{N-1}} s_j^{\sigma_1} \ldots s_j^{\sigma_N}.$$

Note that if $\sigma_i = \emptyset$, then $s_j^{\sigma_i} = s_j^{\emptyset} = s_j^{\{j\}} = \theta$, and thus for all $j \in [n]$,

$$s_j^\sigma = \frac{\theta^{N-|I|}}{\theta^{N-1}} \prod_{i \in I} s_j^{\sigma_i} = \frac{1}{\theta^{|I|-1}} \prod_{i \in I} s_j^{\sigma_i}.$$

The fact that $s_j^{\sigma_i}$ has the same value for every $j \in [n] \setminus \tau_i$ is a direct consequence of Theorem SM1.5 since τ_i is simply-added onto $[n] \setminus \tau_i$.

Finally, to prove the last statements about the signs of $s_j^{\sigma_i}$, observe that for $j \in \tau_i$, the values of $\operatorname{sgn} s_j^{\sigma_i}$ are fully determined by Theorem SM1.2 (sign conditions) since $\sigma_i \in \mathrm{FP}(G|_{\tau_i})$ by hypothesis. In particular, if $\sigma_i \in S_{\tau_i}$, then σ_i survives the addition of every $k \notin \tau_i$, and so $\operatorname{sgn} s_k^{\sigma_i} = -\operatorname{idx}(\sigma_i)$ by Theorem SM1.2 (sign conditions). On the other hand, if $\sigma_i \in D_{\tau_i}$ then σ_i dies in G and so there is some $k \notin \tau_i$ for which $\operatorname{sgn} s_k^{\sigma_i} = \operatorname{idx}(\sigma_i)$. But by the first part of the theorem, all the $s_k^{\sigma_i}$ values are identical for $k \in [n] \setminus \tau_i$, and thus $\operatorname{sgn} s_k^{\sigma_i} = \operatorname{idx}(\sigma_i)$ for all such k. ■

With Lemma SM1.13, it is now straightforward to prove Theorem 3.8 (reprinted below). This theorem generalizes Theorem SM1.8, characterizing every element of $\mathrm{FP}(G)$ in terms of the sets of surviving and dying component fixed points supports, S_{τ_i} and D_{τ_i}. Notice that in the statement of Theorem 3.8, all the fixed point supports of type (a) have the form $\bigcup_{i \in I} \sigma_i$ for $\sigma_i \in S_{\tau_i}$ and $I \subseteq [N]$, while those of type (b) have the form $\bigcup_{i=1}^N \sigma_i$ for $\sigma_i \in D_{\tau_i}$.

Theorem 3.8. *Suppose G has a strongly simply-embedded partition $\{\tau_1|\ldots|\tau_N\}$, and let $\sigma_i \stackrel{\text{def}}{=} \sigma \cap \tau_i$ for any $\sigma \subseteq [n]$. Then $\sigma \in \mathrm{FP}(G)$ if and only if $\sigma_i \in \mathrm{FP}(G|_{\tau_i}) \cup \{\emptyset\}$ for each $i \in [N]$, and either*

(a) *every σ_i is in $\mathrm{FP}(G) \cup \{\emptyset\}$, or*

(b) *none of the σ_i are in $\mathrm{FP}(G) \cup \{\emptyset\}$.*

In other words, $\sigma \in \mathrm{FP}(G)$ if and only if σ is either a union of surviving fixed points σ_i, at most one per component, or it is a union of dying fixed points, exactly one from every component.

Proof. First notice that since G has a strongly simply-embedded partition $\{\tau_1|\ldots|\tau_N\}$, by Lemma SM1.13, for all $j \in [n]$, we have

$$s_j^\sigma = \prod_{i \in I} s_j^{\sigma_i}$$

where $I \stackrel{\text{def}}{=} \{i \mid \sigma_i \neq \emptyset\}$, and we have set $\theta = 1$, without loss of generality. Moreover, $s_j^{\sigma_i}$ is constant across $j \in [n] \setminus \tau_i$ for each $i \in [N]$.

($\Rightarrow$) Suppose $\sigma \in \mathrm{FP}(G)$. Since G has a simply-embedded partition, Theorem 1.4 (menu) guarantees $\sigma_i \in \mathrm{FP}(G|_{\tau_i})$ for every $i \in I$. Thus we can use the values of $\operatorname{sgn} s_j^{\sigma_i}$ given in Lemma SM1.13 to examine the sign conditions for σ. For any $j \in \sigma$, there exists $i \in I$ such that $j \in \sigma_i$, and then

$$\operatorname{sgn} s_j^\sigma = \operatorname{idx}(\sigma_i) \prod_{\{a \in I \setminus \{i\} \ | \ \sigma_a \in S_a\}} -\operatorname{idx}(\sigma_a) \prod_{\{b \in I \setminus \{i\} \ | \ \sigma_b \in D_b\}} \operatorname{idx}(\sigma_b) = (-1)^{|\mathcal{S} \setminus \{i\}|} \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell), \tag{SM1.5}$$

where $\mathcal{S} \stackrel{\text{def}}{=} \{a \in I \mid \sigma_a \in S_a\}$.

Now, observe that if σ contained a mix of $\sigma_a \in S_a$ and $\sigma_b \in D_b$, then there would be $i, j \in \sigma$ such that $i \in \sigma_a$ for some $a \in \mathcal{S}$, while $j \in \sigma_b$ for some $b \notin \mathcal{S}$. In this case,

$$\operatorname{sgn} s_i^\sigma = (-1)^{|\mathcal{S}|-1} \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell) = -(-1)^{|\mathcal{S}|} \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell) = -\operatorname{sgn} s_j^\sigma.$$

But by Theorem SM1.2 (sign conditions), $\sigma \in \operatorname{FP}(G)$ implies that $\operatorname{sgn} s_i^\sigma = \operatorname{sgn} s_j^\sigma$ for all $i, j \in \sigma$, yielding a contradiction. Thus, we must have either $\sigma_i \in S_{\tau_i}$ for all $i \in I$, as in (a), or $\sigma_i \in D_{\tau_i}$ for all $i \in I$ as in (b).

Next we show that in case (b) when $\sigma_i \in D_{\tau_i}$ for all $i \in I$, we must have $I = [N]$, so that σ takes a dying fixed point from every component. Assume to the contrary that $I \subsetneq [N]$ so that there is some $m \in [N]$ such that $\tau_m \cap \sigma = \emptyset$. Then, for $k \in \tau_m$ (so $k \notin \sigma$), we have $\operatorname{sgn} s_k^{\sigma_\ell} = \operatorname{idx}(\sigma_\ell)$ for all $\ell \in I$, by Lemma SM1.13, since $\sigma_\ell \in D_{\tau_\ell}$. Thus

$$\operatorname{sgn} s_k^\sigma = \prod_{\ell \in I} \operatorname{sgn} s_k^{\sigma_\ell} = \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell).$$

Meanwhile, for all $j \in \sigma$ we have $j \in \tau_i$ for some $i \in I$, and Equation (SM1.5) gives

$$\operatorname{sgn} s_j^\sigma = (-1)^{|\mathcal{S} \setminus \{i\}|} \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell) = \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell)$$

since $\mathcal{S} = \emptyset$ because $\sigma_\ell \in D_{\tau_\ell}$ for all $\ell \in I$. Thus,

$$\operatorname{sgn} s_k^\sigma = \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell) = \operatorname{sgn} s_j^\sigma$$

for some $j \in \sigma$ and $k \notin \sigma$, contradicting the sign conditions for $\sigma \in \operatorname{FP}(G)$. Therefore, we must have $I = [N]$.

($\Leftarrow$) First consider case (a) where $\sigma_i \in S_{\tau_i}$ for all $i \in I$. We will show that $\sigma \overset{\text{def}}{=} \bigcup_{i \in I} \sigma_i \in \operatorname{FP}(G)$ by checking the sign conditions. For any $j \in \sigma$, there exists $i \in I$ such that $j \in \tau_i$. Then by Equation (SM1.5), we have

$$\operatorname{sgn} s_j^\sigma = (-1)^{|\mathcal{S} \setminus \{i\}|} \prod_{\ell \in I} \operatorname{idx}(\sigma_\ell) = (-1)^{|I|-1} \prod_{\ell \in I} \operatorname{idx} \sigma_\ell,$$

since $\mathcal{S} = I$ in this case. On the other hand, for $k \notin \sigma$, we have $\operatorname{sgn} s_k^{\sigma_\ell} = -\operatorname{idx} \sigma_\ell$ for all $\ell \in I$, by Lemma SM1.13, since $\sigma_\ell \in S_{\tau_\ell}$. Thus

$$\operatorname{sgn} s_k^\sigma = \prod_{\ell \in I} (-\operatorname{idx} \sigma_\ell) = (-1)^{|I|} \prod_{\ell \in I} \operatorname{idx} \sigma_\ell = -\operatorname{sgn} s_j^\sigma.$$

Therefore $\sigma \in \operatorname{FP}(G)$ by Theorem SM1.2 (sign conditions).

Next, consider case (b) where $\sigma_\ell \in D_{\tau_\ell}$ for all $\ell \in [N]$ (so $I = [N]$). Then for any $j \in \sigma$, there is $i \in [N]$ such that $j \in \sigma_i$ and by Equation (SM1.5), we have

$$\operatorname{sgn} s_j^\sigma = (-1)^{|\mathcal{S} \setminus \{i\}|} \prod_{\ell \in [N]} \operatorname{idx}(\sigma_\ell) = \prod_{\ell \in [N]} \operatorname{idx}(\sigma_\ell),$$

since $\mathcal{S} = \emptyset$. Meanwhile, for any $k \notin \sigma$ there is some m such that $k \in \tau_m$ with $\tau_m \cap \sigma \neq \emptyset$ (since $I = [N]$). Since $\sigma_m \in \mathrm{FP}(G|_{\tau_m})$, we have $\operatorname{sgn} s_k^{\sigma_m} = -\operatorname{idx}(\sigma_m)$ and thus

$$\operatorname{sgn} s_k^{\sigma} = \operatorname{sgn} s_k^{\sigma_m} \prod_{\ell \in [N] \setminus \{m\}} \operatorname{sgn} s_k^{\sigma_\ell} = -\operatorname{idx}(\sigma_m) \prod_{\ell \in [N] \setminus \{m\}} \operatorname{idx}(\sigma_\ell) = -\prod_{\ell \in [N]} \operatorname{idx}(\sigma_\ell) = -\operatorname{sgn} s_j^{\sigma}.$$

Thus sign conditions are satisfied, and so $\sigma \in \mathrm{FP}(G)$. ■

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