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Superspace geometrical realization of the *N*-extended super Virasoro algebra and its dual [1]

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Abstract

We derive properties of *N*-extended $\mathcal{GR}$ super Virasoro algebras. These include adding central extensions, identification of all primary fields and the action of the adjoint representation on its dual. The final result suggest identification with the spectrum of fields in supergravity theories and superstring/M-theory constructed from NSR *N*-extended supersymmetric $\mathcal{GR}$ Virasoro algebras. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently super-derivations were introduced [1] that extend previous work [2] on 1D, $\aleph_0$-extended superspace. This set of super-derivations is closed under graded commutation and contains a super Virasoro-like sub-algebra for all values of *N*-extended supersymmetry. The smallest set of the derivations that forms a closed algebra under the action of the graded commutator contains the following:

$$
\begin{aligned}
&G_{\mathscr{A}}{}^{I} \equiv i\,\tau^{\mathscr{A}+1/2}\left[\partial^{I} - i\,2\,\zeta^{I}\partial_{\tau}\right] \\
&\qquad + 2\left(\mathscr{A}+\tfrac{1}{2}\right)\tau^{\mathscr{A}-1/2}\zeta^{I}\zeta^{K}\partial_{K}\,, \\
&L_{\mathscr{A}} \equiv -\left[\tau^{\mathscr{A}+1}\partial_{\tau} + \tfrac{1}{2}\left(\mathscr{A}+1\right)\tau^{\mathscr{A}}\zeta^{I}\partial_{I}\right], \\
&T^{IJ}_{\mathscr{A}} \equiv \tau^{\mathscr{A}}\left[\zeta^{I}\partial^{J} - \zeta^{J}\partial^{I}\right], \\
&U^{I_1\cdots I_q}_{\mathscr{A}} \equiv i\,(i)^{[\frac{q}{2}]}\tau^{(\mathscr{A}-\frac{(q-2)}{2})}\zeta^{I_1}\cdots\zeta^{I_{q-1}}\partial^{I_q}, \\
&\quad q = 3,\ldots,N+1, \\
&R^{I_1\cdots I_p}_{\mathscr{A}} \equiv (i)^{[\frac{p}{2}]}\tau^{(\mathscr{A}-\frac{(p-2)}{2})}\zeta^{I_1}\cdots\zeta^{I_p}\partial_{\tau}, \\
&\quad p = 2,\ldots,N, \qquad (1)
\end{aligned}
$$

for any number *N* of supersymmetries. (Our notational conventions can be found in [1].) These derivations do not depend on a specific value of *N* and can

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therefore be used for the entire 1D, $\aleph_0$ superspace. For low values of N, not all of the generators appear. For example, $T_{\mathscr{A}}^{IJ}$ and $R_{\mathscr{A}}^{I_1\cdots I_p}$ only appear for superspaces with $N\geq 2$. Generically, $U_{\mathscr{A}}^{I_1\cdots I_q}$ only appears for superspaces with $N\geq 3$. The indices denoted by $\mathscr{A}$, $\mathscr{B}$, etc. denote the level or mode number of the operators. These types of indices take their values in $Z+\frac{1}{2}\oplus Z$.

One of the tasks of this paper is to centrally extend the algebra generated by the above generators. We impose the Jacobi identity on all possible combinations of the generators and find that the centrally extended algebra is given by [5]

$$\begin{aligned}
&\{L_{\mathscr{A}}, L_{\mathscr{B}}\}\\
&\quad=(\mathscr{A}-\mathscr{B})\,L_{\mathscr{A}+\mathscr{B}}+\tfrac{1}{8}c\,(\mathscr{A}^3-\mathscr{A})\,\delta_{\mathscr{A}+\mathscr{B},0}\,,\\
&\{L_{\mathscr{A}}, U_{\mathscr{B}}^{I_1\cdots I_m}\}=-\left[\mathscr{B}+\tfrac{1}{2}(m-2)\,\mathscr{A}\right]U_{\mathscr{A}+\mathscr{B}}^{I_1\cdots I_m},\\
&\{G_{\mathscr{A}}{}^{I}, G_{\mathscr{B}}{}^{J}\}=-i4\,\delta^{IJ}L_{\mathscr{A}+\mathscr{B}}-i2(\mathscr{A}-\mathscr{B})\\
&\qquad\times\left[T_{\mathscr{A}+\mathscr{B}}^{IJ}+2(\mathscr{A}+\mathscr{B})U_{\mathscr{A}+\mathscr{B}\,K}^{IJK}\right]\\
&\qquad-ic\left(\mathscr{A}^2-\tfrac{1}{4}\right)\delta_{\mathscr{A}+B,0}\,\delta^{IJ},\\
&\{L_{\mathscr{A}}, G_{\mathscr{B}}{}^{I}\}=\left(\tfrac{1}{2}\mathscr{A}-\mathscr{B}\right)G_{\mathscr{A}+\mathscr{B}}{}^{I},\\
&\{L_{\mathscr{A}}, R_{\mathscr{B}}^{I_1\cdots I_m}\}=-\left[\mathscr{B}+\tfrac{1}{2}(m-2)\,\mathscr{A}\right]R_{\mathscr{A}+\mathscr{B}}^{I_1\cdots I_m}\\
&\qquad-\left[\tfrac{1}{2}\mathscr{A}(\mathscr{A}+1)\right]U_{\mathscr{A}+\mathscr{B}}^{I_1\cdots I_m J}{}_{J},\\
&\{L_{\mathscr{A}}, T_{\mathscr{B}}^{IJ}\}=-\mathscr{B}\,T_{\mathscr{A}+B}^{IJ},\\
&\{R_{\mathscr{A}}^{I_1\cdots I_m}, R_{\mathscr{B}}^{J_1\cdots J_n}\}\\
&\quad=-(i)^{\sigma(mn)}\left[\mathscr{A}-\mathscr{B}-\tfrac{1}{2}(m-n)\right]\\
&\qquad\times R_{\mathscr{A}+\mathscr{B}}^{I_1\cdots I_m J_1 n\cdots J_n},\\
&\{T_{\mathscr{A}}^{IJ}, T_{\mathscr{B}}^{KL}\}\\
&\quad=T_{\mathscr{A}+B}^{IK}\,\delta^{JL}+T_{\mathscr{A}+B}^{JL}\,\delta^{IK}-T_{\mathscr{A}+B}^{IL}\,\delta^{JK}\\
&\qquad-T_{\mathscr{A}+B}^{JK}\,\delta^{IL}+\tilde{c}\,(\mathscr{A}-\mathscr{B})\\
&\qquad\times(\delta^{IK}\delta^{JL}-\delta^{IL}\delta^{JK})\,\delta_{\mathscr{A}+\mathscr{B},0}\,,
\end{aligned}$$

$$\begin{aligned}
&\{G_{\mathscr{A}}{}^{I}, R_{\mathscr{B}}^{J_1\cdots J_m}\}\\
&\quad=2(i)^{\sigma(m)}\left[\mathscr{B}+(m-1)\,\mathscr{A}+\tfrac{1}{2}\right]R_{\mathscr{A}+\mathscr{B}}^{IJ_1\cdots J_m}\\
&\qquad-(i)^{\sigma(m)}\sum_{r=1}^{m}(-1)^{r-1}\delta^{IJ_r}R_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_{r-1}J_{r+1}\cdots J_m}\\
&\qquad-(-i)^{\sigma(m)}\left[\mathscr{A}+\tfrac{1}{2}\right]U_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_m I}\\
&\qquad+2(i)^{\sigma(m)}\left[\mathscr{A}^2-\tfrac{1}{4}\right]U_{\mathscr{A}+\mathscr{B}}^{IJ_1\cdots J_m K}{}_{K},\\
&\{G_{\mathscr{A}}{}^{I}, U_{\mathscr{B}}^{J_1\cdots J_m}\}\\
&\quad=2(i)^{\sigma(m)}\left[\mathscr{B}+(m-2)\,\mathscr{A}\right]U_{\mathscr{A}+\mathscr{B}}^{IJ_1\cdots J_m}\\
&\qquad-2(-i)^{\sigma(m)}\left[\mathscr{A}+\tfrac{1}{2}\right]\delta^{IJ_m}U_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_{m-1}K}{}_{K}\\
&\qquad-(i)^{\sigma(m)}\sum_{r=1}^{m-1}(-1)^{r-1}\delta^{IJ_r}\\
&\qquad\times U_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_{r-1}J_{r+1}\cdots J_m}\\
&\qquad+2(-i)^{\sigma(m)}\delta^{IJ_m}R_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_{m-1}},\\
&\{R_{\mathscr{A}}^{I_1\cdots I_m}, U_{\mathscr{B}}^{J_1\cdots J_n}\}\\
&\quad=(-i)^{\sigma(mn)}\sum_{r=1}^{m}(-1)^{r-1}\delta^{I_rJ_n}\\
&\qquad\times R_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_{n-1}I_1\cdots I_{r-1}I_{r+1}\cdots I_m}\\
&\qquad+i(i)^{\sigma(mn)}\left[\mathscr{B}-\tfrac{1}{2}(n-2)\right]\\
&\qquad\times U_{\mathscr{A}+\mathscr{B}}^{I_1\cdots I_m J_1\cdots J_n},\\
&\{U_{\mathscr{A}}^{I_1\cdots I_m}, U_{\mathscr{B}}^{J_1\cdots J_n}\}\\
&\quad=-(i)^{\sigma(mn)}\left\{\sum_{r=1}^{m}(-1)^{r-1}\delta^{I_mJ_r}\right.\\
&\qquad\times U_{\mathscr{A}+\mathscr{B}}^{I_1\cdots I_{m-1}J_1\cdots J_{r-1}J_{r+1}\cdots J_{n-1}J_n}\\
&\qquad-(-1)^{mn}\sum_{r=1}^{m}(-1)^{r-1}\delta^{I_rJ_n}\\
&\qquad\left.U_{\mathscr{A}+\mathscr{B}}^{J_1\cdots J_{n-1}I_1\cdots I_{r-1}I_{r+1}\cdots I_{m-1}I_m}\right\},\\
&\{T_{\mathscr{A}}^{IJ}, G_{\mathscr{B}}{}^{K}\}=2\left(\delta^{JK}G_{\mathscr{A}+B}{}^{I}-\delta^{IK}G_{\mathscr{A}+B}{}^{J}\right)\\
&\qquad+2\mathscr{A}\left(\delta^{JK}\,U_{\mathscr{A}+\mathscr{B}\,L}^{I,L}-\delta^{IK}\,U_{\mathscr{A}+\mathscr{B}\,L}^{JL}\right.\\
&\qquad\left.+U_{\mathscr{A}+\mathscr{B}}^{JKI}-U_{\mathscr{A}+\mathscr{B}}^{IKJ}\right),
\end{aligned}$$

[5] Some of the results here contain minor corrections to those given in [1].

$$
\begin{aligned}
&\left[T_{\mathscr{A}}^{IJ}, R_{\mathscr{B}}^{I_1\cdots I_p}\right\} \\
&\quad = \sum_{r=1}^{p} (-1)^{r+1}\left(\delta^{JI_r} R_{\mathscr{A}+B}^{II_1\cdots I_{r-1}I_{r+1}\cdots I_p}\right. \\
&\quad \left. - \delta^{II_r} R_{\mathscr{A}+B}^{JI_1\cdots I_{r-1}I_{r+1}\cdots I_p}\right) + i(-1)^p \\
&\quad \times \mathscr{A}\left(U_{\mathscr{A}+B-1}^{II_1\cdots I_pJ} - U_{\mathscr{A}+B-1}^{JI_1\cdots I_pI}\right), \\
&\left[T_{\mathscr{A}}^{IJ}, U_{\mathscr{B}}^{I_1\cdots I_p}\right\} \\
&\quad = \sum_{r=1}^{p} (-1)^{r+1}\left(\delta^{JI_r} U_{\mathscr{A}+B}^{II_1\cdots I_{r-1}I_{r+1}\cdots I_p}\right. \\
&\quad \left. - \delta^{II_r} U_{\mathscr{A}+B}^{JI_1\cdots I_{r-1}I_{r+1}\cdots I_p}\right) \\
&\quad + \left(\delta^{I_pI} U_{\mathscr{A}+B-1}^{I_1\cdots I_{p-1}J} - \delta^{I_pJ} U_{\mathscr{A}+B-1}^{I_1\cdots I_{p-1}I}\right), \qquad (2)
\end{aligned}
$$

where the function $\sigma(m)=0$ if m is even and -1 if m is odd. Here the central extensions c and $\tilde{c}$ are unrelated since we have only imposed the Jacobi identity. New constraints will arise when we restrict to unitary representations. The algebra exhibits interesting properties such as a generalization of the $\mathrm{SO}(N)$ generators due to the presence of the U and R type fields. The nature of these fields will be discussed throughout as we derive transformation laws.

One way to understand the operators that arise in this new algebra is through methods used to study other infinite dimensional algebras. In particular, we will borrow techniques from coadjoint representation to help interpret these new generators. The coadjoint representation for infinite dimensional algebras [3,6] has appeared in the string theory literature for some time. Its uses include the study of chiral anomalies [10–12], geometric quantization of the Virasoro group [7], the study of orthogonal field theories [4,5,8] and recently in relation to AdS_3 quantum gravity [9].

In this paper we will examine the coadjoint representation of the superspace geometrical representation (''$\mathscr{GR}$'') of the extended super Virasoro algebras as well as some other properties. Although the algebra is quite complex, the coadjoint representation of this particular algebra can generalize many of the above mentioned aspects as well as shed light on the meaning of the new generators and the spectrum of states that may appear in a supergravity or superstring/M-theory based on this algebra.

2. Primary fields

Before going into the coadjoint representation, we would like to identify the primary fields associated with this algebra and their associated conformal weights. Since $\mathscr{L}$ is the generator of diffeomorphisms we can use its action on the other generators to determine the tensor properties of the fields. Let

$$
\mathscr{L}' = \left(L_\xi, G_{\chi^I}{}^I, T_{t^{JK}}^{JK}, \oplus_{\{I_q\}} U_{\mu}^{\{I_q\}}{}_{\{I_q\}}, \oplus_{\{J_q\}} R_{r\{J_q\}}^{\{J_q\}}; \alpha\right), \qquad (3)
$$

represent the generators with generic functions and $\oplus_{\{I_q\}}$ represents the direct sum over all distinct generators. Then from the algebra we see that

$$
\begin{aligned}
&\left[(L_\xi,\alpha),(L_\zeta,\beta)\right\} \\
&\quad = \left(L_{\xi'\zeta-\xi\zeta'}, \frac{c}{i2\pi}\int(\xi''\zeta'-\zeta''\xi')\,dx\right), \\
&\left[L_\xi, G_\chi^I\right\} = G^I_{(-\xi(\chi^I)'+\frac{1}{2}\xi'\chi^I)}, \\
&\left[L_\xi, T_{t^{RS}}^{RS}\right\} = T^{RS}_{(-\xi(t^{RS})')}, \\
&\left[L_\xi, U_{w^{\{V_r\}}}^{\{V_r\}}\right\} = U^{\{V_r\}}_{(-\xi(w^{\{V_r\}})'-\frac{1}{2}(r-2)\xi' w^{\{V_r\}})}, \\
&\left[L_\xi, R_{\rho^{\{T_r\}}}^{\{T_r\}}\right\} = R^{\{T_r\}}_{(-(\rho^{\{T_r\}})'\xi-\frac{1}{2}(r-2)\xi'(\rho^{\{T_r\}}))} \\
&\qquad - \frac{i}{2} U^{\{T_r\}}_{(\xi''\rho^{\{T_r\}})}, \qquad (4)
\end{aligned}
$$

determines the transformation laws of the functions. In the above, we have suppressed the Grassman indices. For example $w^{\bar{V}_1\cdots\bar{V}_n}$ the function associated with the U generators may be written as $w^{\{V_m\}}$ or simply as w. From the coefficient of the ξ' summand in the transformation laws we can write down the conformal weight which is also the rank of the one dimensional tensors. The quantity ξ is a rank one contravariant tensor field, χ^I is a spin half field and t^{RS} is a scalar field. This is to be expected from these fields. However, notice that $w^{\{V_r\}}$ transforms

with conformal weight $\frac{1}{2}(r-2)$ where r takes values from 3 to N when there are N supersymmetries which corresponds to a tower of $N-2$ fields. The quantity $\rho^{\{T_r\}}$ to appears to transform as a rank $\frac{1}{2}(r-2)$ tensor modulo the inhomogeneous term. However, it is this inhomogeneous term that keeps these fields from transforming like tensors. Since the transformations of $w^{\{V_r\}}$ and $\rho^{\{T_r\}}$ are entangled, a natural question to ask is what linear combination of generators produces tensors or in the language of conformal field theory which generators are primary.

To answer this let us consider the generators

$$\mathscr{Q}_{\mathscr{A}}^{I_1\cdots I_p} = \tau^{\mathscr{A}}\zeta^{I_1}\cdots\zeta^{I_p}\partial_\tau\,,$$
$$\mathscr{P}_{\mathscr{A}}^{I_1\cdots I_{p+1}} = \tau^{\mathscr{A}}\zeta^{I_1}\cdots\zeta^{I_p}\partial^{I_{p+1}}\,. \tag{5}$$

These form a closed algebra among themselves and are used to facilitate the computations below. We can write the previous generators as

$$L_{\mathscr{A}} = -\mathscr{Q}_{\mathscr{A}+1} - \tfrac{1}{2}(\mathscr{A}+1)\mathscr{P}^I_{\mathscr{A}\,I}\,,$$
$$G^I_{\mathscr{A}} = i\mathscr{P}^I_{\mathscr{A}+\frac{1}{2}} + 2\mathscr{Q}^I_{\mathscr{A}+\frac{1}{2}} + 2\left(\mathscr{A}+\frac{1}{2}\right)\mathscr{P}^{IK}_{\mathscr{A}-\frac{1}{2}\,K}\,,$$
$$T^{IJ}_{\mathscr{A}} = \mathscr{P}^{IJ}_{\mathscr{A}} - \mathscr{P}^{JI}_{\mathscr{A}}\,,$$
$$R^{I_1\cdots I_p}_{\mathscr{A}} = i^{[\frac{p}{2}]}\mathscr{Q}^{I_1\cdots I_p}_{\mathscr{A}-\frac{p}{2}+1}\,,\quad (p=2,\ldots,N)\,,$$
$$U^{I_1\cdots I_q}_{\mathscr{A}} = i\,(i)^{[\frac{q}{2}]}\mathscr{P}^{I_1\cdots I_q}_{\mathscr{A}-\frac{p}{2}+1}\,,\quad (q=3,\ldots,N+1)\,. \tag{6}$$

Let $\mathscr{F}_{\mathscr{A}}^{I_1\cdots I_p}$ be a primary generator. Then by definition for some particular mode dependent λ this generator satisfies

$$\left[L_{\mathscr{A}}, \mathscr{F}_{\mathscr{B}}^{I_1\cdots I_p}\right] = -\lambda(\mathscr{A},B,p)\mathscr{F}_{\mathscr{A}+B}^{I_1\cdots I_p}\,, \tag{7}$$

for fixed number of indices p. For each value of p (assuming that p is greater than 2), $\mathscr{F}_{\mathscr{A}}^{I_1\cdots I_p}$ can be generically written as

$$\mathscr{F}_{\mathscr{A}}^{I_1\cdots I_p} = c_0(\mathscr{A}+1)\mathscr{Q}_{A+1}^{I_1\cdots I_p} + c_1(\mathscr{A}+1)\mathscr{P}_{\mathscr{A}+1}^{I_1\cdots I_p} + c_2(\mathscr{A})\mathscr{P}_{\mathscr{A}}^{I_1\cdots I_p L}{}_L\,, \tag{8}$$

which gives us three possible mode dependent coefficients to compute, viz c_0, c_1, and c_2. From the commutation relations of $\mathscr{Q}_{\mathscr{A}}^{I_1\cdots I_p}$ and $\mathscr{P}_{\mathscr{A}}^{I_1\cdots I_p}$ the conditions for a primary generator are

$$\lambda c_0(\mathscr{A}+B+1) = c_0(\mathscr{B}+1) \times\left[(\mathscr{B}-A)+\tfrac{1}{2}p(\mathscr{A}+1)\right],$$
$$\lambda c_1(\mathscr{A}+B+1) = c_1(\mathscr{B}+1) \times\left[(\mathscr{B}-A)+\tfrac{1}{2}p(\mathscr{A}+1)\right],$$
$$\lambda c_2(\mathscr{A}+B) = -c_0(\mathscr{B}+1)\left[\frac{p}{2}(\mathscr{A}+1)\right]. \tag{9}$$

There are three classes of solutions.

1. Class 1:
   Setting $c_0=1$, $c_1=0$ and $c_2(\mathscr{A}) = a_1\mathscr{A}+a_2$ we find that
   $$\mathscr{F}^{1\,I_1\cdots I_p}_{\mathscr{B}} = \mathscr{Q}^{I_1\cdots I_p}_{\mathscr{B}+1} + \left(\frac{\mathscr{B}+1}{2-p}\right)\mathscr{P}^{I_1\cdots I_p L}_{\mathscr{B}}{}_L\,, \tag{10}$$
   is a primary field with $\lambda = (\mathscr{B}+\frac{p}{2}) - \mathscr{A}(1-\frac{p}{2})$. This can be rewritten in terms of the original generators as
   $$R^{I_1\cdots I_p}_{\mathscr{B}+\frac{p}{2}} - i\left(\frac{\mathscr{B}+1}{2-p}\right)U^{I_1\cdots I_p^L}_{\mathscr{B}+\frac{p}{2}-1\,L}\,,\quad (p\neq 2) \tag{11}$$
   is a primary field.
2. Class 2:
   Setting $c_0=0$, $c_1=1$ and $c_2(\mathscr{A}) = a_1\mathscr{A}+a_2$ we find that
   $$\mathscr{F}^{2\,IJ}_{\mathscr{B}} = \mathscr{P}^{IJ}_{\mathscr{B}+1} + (\mathscr{B}+1)\,\mathscr{P}^{IJK}_{\mathscr{B}}{}_K\,. \tag{12}$$
   In this case $p=2$ was forced as a condition, thus the above λ simplifies to $\lambda = \mathscr{B}+1$.
3. Class 3:
   Setting $c_2=0$, $c_1=1$ and $c_0(\mathscr{A}) = a_1\mathscr{A}+a_2$ we find that
   $$\mathscr{F}^{3\,I_1\cdots I_p}_{\mathscr{B}} = \mathscr{P}^{I_1\cdots I_p}_{\mathscr{B}+1}\,, \tag{13}$$
   is a primary field. In this case $c_0=0$ was forced as a condition.

From these three solutions we can find all the primary fields of the original algebra.

$$L_{\mathscr{B}} = -\mathscr{F}^{1}_{\mathscr{B}} \quad (p=0),$$

$$G^{I}_{\mathscr{B}} = 2\mathscr{F}^{1I}_{\mathscr{B}-\frac{1}{2}} + i\mathscr{F}^{3I}_{\mathscr{B}-\frac{1}{2}} \quad (p=1),$$

$$T^{IJ}_{\mathscr{B}} = \mathscr{F}^{3IJ}_{\mathscr{B}-1} - \mathscr{F}^{3JI}_{\mathscr{B}-1} \quad (p=2),$$

$$U^{I_1\cdots pI_p}_{\mathscr{B}+\frac{p}{2}} = ii^{[\frac{p}{2}]}\mathscr{F}^{3I_1\cdots I_p}_{\mathscr{B}} \quad (p\geq 3),$$

$$R^{I_1\cdots pI_p}_{\mathscr{B}+\frac{p}{2}} - i\left(\frac{\mathscr{B}+1}{2-p}\right)U_{\mathscr{B}+\frac{p}{2}-1\,L}{}^{I_1\cdots I_p L} = (i)^{[\frac{p}{2}]}\mathscr{F}^{1I_1\cdots I_p}_{\mathscr{B}} \quad (p\geq 3). \tag{14}$$

We note that $\mathscr{R}^{IJ}_{\mathscr{A}}$ for no value of $\mathscr{A}$ admits a primary field. Stated in a slightly different way, the set of generators given by

$$G_{\mathscr{A}}{}^{I} \equiv i\tau^{\mathscr{A}+\frac{1}{2}}\left[\partial^{I} - i2\zeta^{I}\partial_{\tau}\right] + 2\left(\mathscr{A}+\frac{1}{2}\right)\tau^{\mathscr{A}-\frac{1}{2}}\zeta^{I}\zeta^{K}\partial_{K},$$

$$L_{\mathscr{A}} \equiv -\left[\tau^{\mathscr{A}+1}\partial_{\tau} + \tfrac{1}{2}(\mathscr{A}+1)\tau^{\mathscr{A}}\zeta^{I}\partial_{I}\right],$$

$$T^{IJ}_{\mathscr{A}} \equiv \tau^{\mathscr{A}}\left[\zeta^{I}\partial^{J} - \zeta^{J}\partial^{I}\right],$$

$$U^{I_1\cdots I_q}_{\mathscr{A}} \equiv i(i)^{[\frac{q}{2}]}\tau^{(\mathscr{A}-\frac{(q-2)}{2})}\zeta^{I_1}\cdots\zeta^{I_{q-1}}\partial^{I_q}, \quad q=3,\ldots,N+1,$$

$$\mathscr{R}^{I_1\cdots I_p}_{\mathscr{A}} \equiv (i)^{[\frac{p}{2}]}\tau^{(\mathscr{A}-\frac{p}{2})}\zeta^{I_1}\cdots\zeta^{I_p} \times\left[\tau\partial_{\tau} + \left(\frac{\mathscr{A}+1}{p-2}\right)\zeta^{L}\partial_{L}\right], \quad p=3,\ldots,N,$$

$$R^{IJ}_{\mathscr{A}} \equiv i\tau^{\mathscr{A}}\zeta^{I}\zeta^{J}\partial_{\tau}, \tag{15}$$

possesses only one non-primary generator, namely $R^{IJ}_{\mathscr{A}}$. We will refer to this basis as the ''almost primary basis'' for the $\mathscr{GR}$ super-Virasoro algebra.

3. The coadjoint representation

In this paper we will examine the coadjoint representation of the superspace geometrical representation of the extended super Virasoro algebras as well as some other properties. Although the algebra is quite complex the coadjoint representation of this particular algebra can generalize many of the above mentioned aspects as well as shed light on the meaning of the new generators in the algebra.

3.1. An example

To begin we will use the semi-direct product of the Virasoro algebra and an affine Lie algebra on the circle to fix the notation and familiarity of the coadjoint representation. In this case we have an affine Lie algebra associated with the loop group G together with the Virasoro algebra given by

$$\left[J^{\alpha}_{N}, J^{\beta}_{M}\right] = if^{\alpha\beta\gamma}J^{\gamma}_{N+M} + Nk\delta_{M+N,0}\delta^{\alpha\beta},$$

$$\left[L_N, J^{\alpha}_{M}\right] = -MJ^{\alpha}_{M+N},$$

$$\left[L_N, L_M\right] = (N-M)L_{N+M} + \frac{\hat{c}}{12}(N^3-N)\delta_{N+M,0}, \tag{16}$$

where $\hat{c} = \frac{2k\mathrm{Dim}(G)}{2k+c_v}$, $\mathrm{Dim}(G)$ is the dimension of the group and c_v is the value of the quadratic Casimir in the adjoint representation. Let $\left(L_A, J^{\beta}_{B}, \rho\right)$ denote a centrally extended adjoint vector. Then from the commutation relations above one may write the adjoint action on the adjoint vectors as

$$\left(L_A, J^{\beta}_{B}, \rho\right) * \left(L_{N'}, J^{\alpha'}_{M'}, \mu\right) = \Big((A-N')L_{A+N'}, -M'J^{\alpha'}_{A+M'} + BJ^{\beta}_{B+N'} + if^{\beta\alpha'\lambda}J^{\lambda}_{B+M'}, \frac{\hat{c}}{12}(A^3-A)\delta_{A+N',0} + Bk\delta^{\alpha'\beta}\delta_{B+M',0}\Big). \tag{17}$$

Now let $\left(\tilde{L}_N, \tilde{J}^{\alpha}_{M}, \tilde{\mu}\right)$ denote an element of the dual space of the algebra and let

$$\left\langle\left(\tilde{L}_N, \tilde{J}^{\alpha}_{M}, \tilde{\mu}\right)\middle|\left(L_{N'}, J^{\alpha'}_{M'}, \mu'\right)\right\rangle = \delta^{N,N'} + \delta^{\alpha,\alpha'}\delta_{M,M'} + \mu\tilde{\mu}, \tag{18}$$

define a suitable pairing. By requiring that this pairing be an invariant under the action of any of the adjoint elements, say $\left(L_A, J^{\beta}_{B}, \rho\right)$, the coadjoint rep-

resentation can be defined. The adjoint action acts as a derivation so that by Leibnitz rule one has

$$\left\langle\left(\tilde{L}_N,\tilde{J}_M^{\alpha},\tilde{\mu}\right)\middle|\left(L_A,J_B^{\beta},\rho\right)*\left(L_{N'},J_{M'}^{\alpha'},\mu\right)\right\rangle = -\left\langle\left(L_A,J_B^{\beta},\rho\right)*\left(\tilde{L}_N,\tilde{J}_M^{\alpha},\tilde{\mu}\right)\middle|\left(L_{N'},J_{M'}^{\alpha'},\mu\right)\right\rangle. \tag{19}$$

Thus the transformation properties of the coadjoint vectors are defined through,

$$\begin{aligned}\left(L_A,J_B^{\beta},\rho\right)*\left(\tilde{L}_N,\tilde{J}_M^{\alpha},\tilde{\mu}\right) = \Bigg(&-(2A-N)\tilde{L}_{N-A}-B\delta^{\alpha\beta}\tilde{L}_{M-B} \\ &-\frac{\tilde{\mu}\hat{c}}{12}(A^3-A)\tilde{L}_{-A}, \\ &(M-A)\tilde{J}_{M-A}^{\alpha}-if^{\beta\nu\alpha}\tilde{J}_{M-B}^{\nu}-\tilde{\mu}Bk\tilde{J}_{-B}^{\beta},0\Bigg).\end{aligned} \tag{20}$$

Instead of using components, let us write $F=(f(\theta),\hat{h}(\theta),a)$ as an arbitrary adjoint vector and $B=(b(\theta),h(\theta),\mu)$ as an arbitrary coadjoint vector, where $f,\hat{h},b$, and h are functions. For the algebra we choose the realization,

$$L_N=i\exp(iN\theta)\partial_\theta, \quad J_N^{\alpha}=\tau^{\alpha}\exp(iN\theta), \tag{21}$$

and normalize the generators so that $Tr(\tau^{\alpha}\tau^{\beta})=\delta^{\alpha\beta}$. Then Eq. (20) may be written as

$$\begin{aligned}\delta_F B \equiv \left(f(\theta),\hat{h}(\theta),a\right)*\left(b(\theta),h(\theta),\mu\right) = -\Bigg(&2f'b+b'f+i\frac{\hat{c}\mu}{12}f'''+Tr[h\hat{h}'], \\ &h'f+hf'+\left[\hat{h}h-h\hat{h}\right]+ik\mu\hat{h}',0\Bigg),\end{aligned} \tag{22}$$

where $'$ denotes ∂_θ. The above equation provides an interpretation of the adjoint elements and coadjoint elements in terms of physical fields in one dimension [4,8]. We already know that the Virasoro sector transforms functions as one dimensional coordinate transformations (up to central extension). For example b transforms as a rank two tensor field in one dimension where the infinitesimal coordinate transformation is given by f. From the second element of the triplet in Eq. (22), on sees that the function h transforms as a one dimensional gauge field with gauge parameter $\hat{h}$. The $h'f+hf'$ contribution to the transformation of h simply shows that the field h transforms as a rank one covariant tensor. The peculiar transformation is the $Tr[h\hat{h}']$ that appears in the transformation of b. This suggests that this rank two tensor can be shifted by fields built purely from the gauge sector. In [4,8] such terms are interpreted as coming from an interaction lagrangian. In any case the relationship between different members of the algebra juxtaposed to the dual space becomes manifest through the coadjoint representation.

Those adjoint vectors, F, that leave B invariant will generate the *isotropy group* for B. Setting Eq. (22) to zero determines the isotropy equation for B. Eq. (22) then determines the tangent space on the orbit of B. Thus for coadjoint elements B_1 and B_2, we may construct the symplectic two form by writing

$$\Omega_B(B_1,B_2)=\langle B|[F_1,F_2]\rangle, \tag{23}$$

where for example $\delta_{F_1}B=B_1$. In [4,8] the equation of isotropy is related to constraint equations that come from a two dimensional field theory.

3.2. N-extended $\mathcal{GR}$ super Virasoro algebra dual space

To proceed we will let $\bar{\mathscr{L}}$ denote a generic coadjoint vector and let $\mathscr{L}$ and $\mathscr{L}'$ denote adjoint vectors. The pairing $\langle\bar{\mathscr{L}}|\mathscr{L}\rangle$ is an invariant so we require $\mathscr{L}'\langle\bar{\mathscr{L}}|\mathscr{L}\rangle=0$. By Leibnitz rule this means that $\langle\mathscr{L}'*\bar{\mathscr{L}}|\mathscr{L}\rangle+\langle\bar{\mathscr{L}}|\mathscr{L}'*\mathscr{L}\rangle=0$. It is from here that we can extract how $\mathscr{L}'*\bar{\mathscr{L}}$ acts. The quantity $\mathscr{L}'*\bar{\mathscr{L}}$ will carry the dual space back into itself. In our notation we will use as a basis for the N-extended Super Virasoro Algebra

$$\begin{aligned}\mathscr{L}' &= \left(L_a,G_b{}^I,T_c{}^{JK},\oplus_{\{I_q\}}U^{\{I_q\}}_{\{d_q\}}\oplus_{\{I_p\}}R^{\{J_q\}}_{\{e_q\}};\alpha\right), \\ \mathscr{L} &= \left(L_z,G_\gamma{}^Q,T_X^{RS},\oplus_{\{V_l\}}U^{\{V_l\}}_{\{w_l\}}\oplus_{\{T_m\}}R^{\{T_m\}}_{\{h_m\}};\beta\right), \\ \bar{\mathscr{L}} &= \Bigg(\bar{L}_{\bar{z}},\bar{G}_{\bar{\gamma}}{}^{\bar{Q}},\bar{T}^{\bar{R}\bar{S}}_{\bar{X}},\oplus_{\{\bar{V}_{\bar{l}}\}}\bar{U}^{\{\bar{V}_{\bar{l}}\}}_{\{\bar{W}_{\bar{l}}\}}, \\ &\qquad \oplus_{\{\bar{T}_m\}}\bar{R}_{\bar{h}_{\bar{m}}}{}^{\{\bar{T}_m\}};\bar{\beta}\Bigg),\end{aligned} \tag{24}$$

We will write a generic functions $f = f_a \tau^a$ and $z = z_b \tau^{b+1/2}$ and realize the Virasoro generators as

$$L_\xi \equiv \xi^{A+1} L_A = -\left[\left(\xi^{A+1}\tau^{A+1}\right)\partial_\tau + \tfrac{1}{2}\xi^{A+1}\tau^A\zeta^I\partial_I\right] = -\left[\xi\partial_\tau + \tfrac{1}{2}\xi\tau^A\zeta^I\partial_I\right]. \quad (25)$$

We will use the subscript of the generators and dual for a generic function of τ. Each generator and dual element will have a specific function and how these functions transform under the action of specific generators is the aim of this paper.

Since the action will come from $\mathscr{L}' * \overline{\mathscr{L}}$ we will denote a generic function as

$$\mathscr{L}' = \left(L_\xi, G_{\chi^I}{}^I, T_t{}^{JK}, \oplus\, U_\mu^{I_1\cdots I_q}{}_{J_1\cdots J_q}, \oplus\, R_r^{J_1\cdots J_q}{}_{J_1\cdots J_q}; \alpha\right),$$

$$\overline{\mathscr{L}} = \left(\overline{L}_D, \overline{G}_{\psi^{\overline{Q}}}^{\overline{Q}}, \overline{T}_\tau^{\overline{R}\overline{S}}, \oplus\, \overline{U}_\omega^{\overline{V}_1\cdots\overline{V}_q}{}_{\overline{V}_1\cdots\overline{V}_q}, \oplus\, \overline{R}_\rho^{\overline{J}_1\cdots\overline{J}_p}{}_{\overline{J}_1\cdots\overline{J}_p}; \overline{\beta}\right). \quad (26)$$

The coadjoint action is quite tedious but we can organize the computation by examining the outcome of each of the commutation relations in the adjoint representation. Table 1 symbolically summarizes our results. In the notation of Table 1 $L * L$ is just the commutator of two arbitrary Virasoro generators while $L * \overline{L} \to \overline{L}$ is the coadjoint action from an application of the Virasoro generator on its dual $\overline{L}$ that maps back into the duals of the Virasoro generators. Multiple entries in the second column correspond to the different coadjoint actions that can be extracted from the commutator in the first column.

From these we can see that $\mathscr{L}' * \overline{\mathscr{L}}$ will lead to changes in the coadjoint vectors as:

$$\begin{aligned}
\delta\overline{L} &= L*\overline{L} + G*\overline{G} + T*\overline{T} + U*\overline{U} + R*\overline{R} + R*\overline{U},\\
\delta\overline{G} &= L*\overline{G} + G*\overline{L} + G*\overline{T} + G*\overline{U} + T*\overline{G} + U*\overline{U} + U*\overline{R} + R*\overline{R} + R*\overline{U},\\
\delta\overline{T} &= L*\overline{T} + G*\overline{G} + G*\overline{U} + T*\overline{T} + U*\overline{U} + R*\overline{R} + R*\overline{U},\\
\delta\overline{U} &= L*\overline{U} + G*\overline{U} + G*\overline{R} + T*\overline{U} + U*\overline{U} + R*\overline{R} + R*\overline{U},\\
\delta\overline{R} &= L*\overline{R} + L*\overline{U} + G*\overline{R} + G*\overline{U} + T*\overline{R} + T*\overline{U} + U*\overline{R} + U*\overline{U} + R*\overline{R},\\
\delta\overline{\beta} &= 0. \qquad (27)
\end{aligned}$$

Table 1

Commutator	Co-adjoint action(s)
$L*L$	$L*\overline{L}\to\overline{L}$
$L*G$	$L*\overline{G}\to\overline{G}$
$L*T$	$L*\overline{T}\to\overline{T}$
$L*U$	$L*\overline{U}\to\overline{U}$
$L*R$	$L*\overline{R}\to\overline{R}$, $L*\overline{U}\to\overline{R}$
$G*L$	$G*\overline{G}\to\overline{L}$
$G*G$	$G*\overline{L}\to\overline{G}$, $G*\overline{T}\to\overline{G}$, $G*\overline{U}\to\overline{G}$
$G*T$	$G*\overline{G}\to\overline{T}$, $G*\overline{U}\to\overline{T}$
$G*U$	$G*\overline{U}\to\overline{U}$, $G*\overline{R}\to\overline{U}$
$G*R$	$G*\overline{R}\to\overline{R}$, $G*\overline{U}\to\overline{R}$
$T*L$	$T*\overline{T}\to\overline{L}$
$T*G$	$T*\overline{G}\to\overline{G}$
$T*T$	$T*\overline{T}\to\overline{T}$
$T*U$	$T*\overline{U}\to\overline{U}$
$T*R$	$T*\overline{R}\to\overline{R}$, $T*\overline{U}\to\overline{R}$
$U*L$	$U*\overline{U}\to\overline{L}$
$U*G$	$U*\overline{U}\to\overline{G}$, $U*\overline{R}\to\overline{G}$
$U*T$	$U*\overline{U}\to\overline{T}$
$U*U$	$U*\overline{U}\to\overline{U}$
$U*R$	$U*\overline{R}\to\overline{R}$, $U*\overline{U}\to\overline{R}$
$R*L$	$R*\overline{R}\to\overline{L}$, $R*\overline{U}\to\overline{L}$
$R*G$	$R*\overline{R}\to\overline{G}$, $R*\overline{U}\to\overline{G}$
$R*T$	$R*\overline{R}\to\overline{T}$, $R*\overline{U}\to\overline{T}$
$R*U$	$R*\overline{R}\to\overline{U}$, $R*\overline{U}\to\overline{U}$
$R*R$	$R*\overline{R}\to\overline{R}$

3.3. Explicit variations

3.3.1. $L * L$ commutator

Starting with the invariant pairing we have:

$$(L_a,\alpha)\left\langle\left(\overline{L}_{\bar{z}},\overline{\beta}\right)|(L_z,\beta)\right\rangle = 0. \quad (28)$$

Then by Leibnitz rule,

$$\left\langle(L_a,\alpha)*\left(\overline{L}_{\bar{z}},\overline{\beta}\right)|(L_z,\beta)\right\rangle + \left\langle\left(\overline{L}_{\bar{z}},\overline{\beta}\right)|(L_a,\alpha)*(L_z,\beta)\right\rangle = 0. \quad (29)$$

Since we know the adjoint action we may write

$$\begin{aligned}&\langle (L_a,\alpha)*\left(\overline{L}_{\bar z},\overline{\beta}\right)|(L_z,\beta)\rangle\\&\quad=-\langle\left(\overline{L}_{\bar z},\overline{\beta}\right)|\big((a-z)\,L_{a+z},\\&\qquad+\tfrac{1}{8}c(a^3-a)\,\delta_{a+z,0}\big)\rangle\end{aligned}\tag{30}$$

which implies that

$$\begin{aligned}&\langle (L_a,\alpha)*\left(\overline{L}_{\bar z},\overline{\beta}\right)|(L_z,\beta)\rangle\\&\quad=-\left\{(a-z)\,\delta_{\bar z,a+z}+\tfrac{1}{8}c(a^3-a)\,\overline{\beta}\,\delta_{a+z,0}\right\}\\&\quad\to(L_a,\alpha)*\left(\overline{L}_{\bar z},\overline{\beta}\right)\\&\quad=-\Big((2a-\bar z)\,\overline{L}_{\bar z-a}\\&\qquad+\tfrac{1}{8}c\,\overline{\beta}\,(a^3-a)\,\overline{L}_{-a},0\Big),\end{aligned}\tag{31}$$

where $z=\bar z-a$, so that

$$\begin{aligned}(L_a,\alpha)*\left(\overline{L}_{\bar z},\overline{\beta}\right)=-\Big((2a-\bar z)\,\overline{L}_{\bar z-a}\\+\tfrac{1}{8}c\,\overline{\beta}\,(a^3-a)\,\overline{L}_{-a},0\Big).\end{aligned}\tag{32}$$

Rewriting in terms of functions instead of modes we have for functions ξ and D,

$$\left(L_\xi,\alpha\right)*\left(\overline{L}_D,\overline{\beta}\right)=\left(\overline{L}_{\tilde D},0\right),\tag{33}$$

where $\tilde D=-(2\xi' D+\xi D'+\frac{1}{8}c\,\overline{\beta}\,\xi''')$. This shows the usual transformation of a quadratic differential, D, with respect to the vector field ξ. Up to the inhomogeneous term D transforms as a rank two tensor. It is the inhomogeneous term that violates tensorality. From the adjoint action one sees that ξ transforms as a rank one contravariant tensor in one dimension making it easy to identify with ξ^α from a Lie derivative. In the same way D can be thought of as a two index object $D_{\alpha\beta}$. This suggests that a spin two type object is present in the spectrum. Throughout we will treat the action of ξ as a one dimensional Lie derivative (up to extensions) in order to understand the type of fields that are present in the dual.

3.3.2. $G*L$ commutator

In the same way we examine the action of the ${G_b}^I$ on the pairing. Since the pairing is invariant we have

$${G_b}^I\langle\,\overline{G}_{\bar y}{}^{\overline{Q}}\,|\,L_z\,\rangle=0,\tag{34}$$

by Leibnitz

$$\langle\,{G_b}^I*\overline{G}_{\bar y}{}^{\overline{Q}}\,|\,L_z\,\rangle+\langle\,\overline{G}_{\bar y}{}^{\overline{Q}}\,|\,{G_b}^I*L_z\,\rangle=0,\tag{35}$$

which implies that

$$\begin{aligned}\langle\,{G_b}^I*\overline{G}_{\bar y}{}^{\overline{Q}}\,|\,L_z\,\rangle&=-\langle\,\overline{G}_{\bar y}{}^{\overline{Q}}\,|-\left(\tfrac{1}{2}z-b\right){G_b}^I\,\rangle\\&=\left(\tfrac{1}{2}z-b\right)\delta^{I\overline{Q}}\,\delta_{\bar y,b+z},\end{aligned}\tag{36}$$

where $z=\bar y-b$. This yields

$${G_b}^I*\overline{G}_{\bar y}{}^{\overline{Q}}=\left(\tfrac{1}{2}\bar y-\tfrac{3}{2}b\right)\overline{L}_{\bar y-b}\,\delta^{\overline{Q}I}.\tag{37}$$

Rewriting in terms of functions we have we have that

$$\begin{aligned}&G^I_{\chi^I}*\overline{G}^{\overline{Q}}_{\Psi^{\overline{Q}}}=\overline{L}_f,\\&\text{where }f=\left(\tfrac{1}{2}\left(\Psi^{\overline{Q}}\right)'\chi^I-\tfrac{3}{2}\left(\chi^I\right)'\Psi^{\overline{Q}}\right)\delta^{\overline{Q}I}.\end{aligned}\tag{38}$$

3.3.3. All $*$ commutators

$$L_\xi*\left(\overline{L}_D,\overline{\beta}\right)=\overline{L}_{\tilde D},$$

$$\tilde D=-2\,\xi' D-\xi\,D'-\frac{c\overline{\beta}}{8}\,\xi''',$$

$$L_\xi*\overline{G}^{\overline{Q}}_{\Psi^{\overline{Q}}}=\overline{G}^{\overline{Q}}_{\Psi^{\tilde{\overline{Q}}}},\quad \Psi^{\tilde{\overline{Q}}}=-\left(\frac{3}{2}\xi'\psi^{\overline{Q}}+\xi\left(\psi^{\overline{Q}}\right)'\right),$$

$$L_\xi*\overline{T}^{\overline{R}\overline{S}}_\tau=\overline{T}^{\overline{R}\overline{S}}_{\tilde\tau},\quad \tilde\tau^{\overline{R}\overline{S}}=-\xi'\tau^{\overline{R}\overline{S}}-\xi\left(\tau^{\overline{R}\overline{S}}\right)',$$

$$\begin{aligned}L_\xi*\overline{U}^{\overline{V}_1\cdots\overline{V}_n}_{\omega^{\overline{V}_1\cdots\overline{V}_n}}&=\overline{U}^{\overline{V}_1\cdots\overline{V}_n}_{\tilde\omega^{\overline{V}_1\cdots\overline{V}_n}}+\frac{i}{2}(i)^{[\frac{n-2}{2}]-[\frac{n}{2}]}\\&\quad\times\overline{R}^{[\overline{V}_1\cdots\overline{V}^{n-2}}_{\xi''\omega^{\overline{V}_1\cdots\overline{V}_n}}\,\delta_{V_{n-1}]V_n},\end{aligned}$$

$$\tilde\omega^{\overline{V}_1\cdots\overline{V}_n}=\left(\frac{n}{2}-2\right)\xi'\,\omega^{\overline{V}_1\cdots\overline{V}_n}-\xi\left(\omega^{\overline{V}_1\cdots\overline{V}_n}\right)',$$

$$L_\xi*\overline{R}^{\overline{T}_1\cdots\overline{T}_m}_{\rho^{\overline{T}_1\cdots\overline{T}_m}}=\overline{R}^{\overline{T}_1\cdots\overline{T}_m}_{\tilde\rho^{\overline{T}_1\cdots\overline{T}_m}},$$

$$\tilde\rho^{\overline{T}_1\cdots\overline{T}_m}=\left(\frac{m}{2}-2\right)\xi'\,\rho^{\overline{T}_1\cdots\overline{T}_m}-\xi\left(\rho^{\overline{T}_1\cdots\overline{T}_m}\right)',$$

$$G^I_{\chi^I}*\overline{G}^{\overline{Q}}_{\psi^{\overline{Q}}}=\delta^{I\overline{Q}}\,\overline{L}_{\tilde\xi}+4\overline{T}^{I\overline{Q}}_{(\chi^I\psi^{\overline{Q}})},$$

$$\tilde\xi=\tfrac{1}{2}\left(\psi^{\overline{Q}}\right)'\chi^I-\tfrac{3}{2}\left(\chi^I\right)'\psi^{\overline{Q}},$$

$$G^I_{\chi^I}*\left(\overline{L}_D,\overline{\beta}\right)=4i\,\overline{G}^I_{(-\chi^I D-\overline{\beta}c(\chi^I)')},$$

$$G^I_{\chi^I}*\overline{T}^{\overline{R}S}_{\overline{R}S}=\frac{i}{2}\left(\overline{G}^S_{\chi^S}\,\delta^{\overline{R}I}-\overline{G}^{\overline{R}}_{\chi^{\overline{R}}}\,\delta^{I\overline{S}}\right),$$

$$\chi^{\bar{R}} * \chi^{S} = 2(\chi^{I})' \tau^{\bar{R}S} + \chi^{I} (\tau^{\bar{R}S})',$$

$$G^{I}_{\chi^{I}} * \bar{R}^{\bar{T}_1 \cdots \bar{T}_m}_{\rho^{\bar{T}_1 \cdots \bar{T}_m}}$$

$$= 2i(i)^{m+1}(i)^{[\frac{m+2}{2}]-[\frac{m}{2}]} \bar{U}^{[\bar{T}_1 \cdots \bar{T}_m]}_{(\chi^I \rho^{\bar{T}_1 \cdots \bar{T}_m})}$$

$$- 2i^{[\frac{m-1}{2}]-[\frac{m-2}{2}]}$$

$$\times \delta^{I[\bar{T}_1} \bar{R}^{\bar{T}_2 \cdots \bar{T}_m]}_{((\chi^I)' \rho^{\bar{T}_1 \cdots \bar{T}_m} - (\chi^I)(\rho^{\bar{T}_1 \cdots \bar{T}_m})')}$$

$$- (i)(i)^{[\frac{m+1}{2}]-[\frac{m}{2}]}$$

$$\times \sum_{r=1}^{m+1} (-1)^{r-1} \bar{R}^{\bar{T}_1 \cdots \bar{T}_{r-1} I \bar{T}_{r+1} \cdots \bar{T}_m}_{(\chi^I \rho^{\bar{T}_1 \cdots \bar{T}_m})},$$

$$T^{JK}_{t} * \bar{G}^{\bar{Q}}_{\psi^{\bar{Q}}} = -2\left(\bar{G}^{K}_{(t^{JK} \psi^{\bar{Q}})} \delta^{\bar{Q}J} - \bar{G}^{J}_{(t^{JK} \psi^{\bar{Q}})} \delta^{\bar{Q}K} \right),$$

$$G^{I}_{\chi^{I}} * \bar{U}^{\bar{V}_1 \cdots \bar{V}_n}_{\omega^{\bar{V}_1 \cdots \bar{V}_n}}$$

$$= -2i^{[\frac{n-1}{2}]-[\frac{n}{2}]}$$

$$\times \delta^{I[\bar{V}_1} \bar{U}^{\bar{V}_2 \cdots \bar{V}_n]}_{((n-4)(\chi^I)' \omega^{\bar{V}_1 \cdots \bar{V}_n} - (\chi^I)(\omega^{\bar{V}_1 \cdots \bar{V}_n})')}$$

$$+ 2(-1)^{n-1}(i)^{[\frac{n-1}{2}]-[\frac{n}{2}]}$$

$$\times \delta^{I[\bar{V}_n} \bar{U}^{\bar{V}_1 \cdots \bar{V}_n]K}_{((\chi^I)' \omega^{\bar{V}_1 \cdots \bar{V}_n})}$$

$$+ (i)(i)^{[\frac{n-1}{2}]-[\frac{n}{2}]} \sum_{r=1}^{n} \bar{U}^{[\bar{V}_1 \cdots \bar{V}_r}_{(\chi^I \omega^{\bar{V}_1 \cdots \bar{V}_n})} \bar{V}_{r+1} \cdots \bar{V}_n \ \delta^{\bar{V}_r]I}$$

$$+ \bar{G}^{[\bar{V}_2}_{(-4i(\chi^I)'(\omega^{\bar{V}_1 \cdots \bar{V}_n)'} - 2i(\chi^I)(\omega^{\bar{V}_1 \cdots \bar{V}_n)''})}$$

$$\times \delta^{V_3 V_4} \delta^{\bar{V}_1]I} \delta^{n4}$$

$$- 2i(-1)^{n}(i)^{[\frac{n}{2}]-[\frac{n-1}{2}]} \delta^{I[\bar{V}_n} \bar{R}^{\bar{V}_1 \cdots \bar{V}^{n-1}]}_{(\chi^I \omega^{\bar{V}_1 \cdots \bar{V}_n})},$$

$$T^{JK}_{t} * \bar{U}^{\bar{V}_1 \cdots \bar{V}_n}_{\omega_{\bar{V}_1 \cdots \bar{V}_n}}$$

$$= -\sum_{r=1}^{n-1} (-1)^{n+1} \Big(\delta^{J[\bar{V}_1} \bar{U}^{\bar{V}_2 \cdots \bar{V}^{r-1}|K|\bar{V}_{r+1} \cdots]\bar{V}_n}_{(t^{JK} \omega^{\bar{V}_1 \cdots \bar{V}_n})}$$

$$- \delta^{K[\bar{V}_1} \bar{U}^{\bar{V}_2 \cdots \bar{V}_{r-1}|J|\bar{V}_{r+1} \cdots]\bar{V}_n}_{(t^{JK} \omega^{\bar{V}_1 \cdots \bar{V}_n})} \Big)$$

$$+ \bar{U}^{[\bar{V}_1 \cdots \bar{V}_{n-1}]J}_{(t^{JK} \omega^{\bar{V}_1 \cdots \bar{V}_n})} \delta^{\bar{V}_n K}$$

$$- \bar{U}^{[\bar{V}_1 \cdots \bar{V}_{n-1}]K}_{(t^{JK} \omega^{\bar{V}_1 \cdots \bar{V}_n})} \delta^{\bar{V}_n J}$$

$$- i(-1)^{n-2} \left(\delta^{K \bar{V}_n} \delta^{J[\bar{V}_1} - \delta^{J \bar{V}_n} \delta^{K[\bar{V}_1} \right)$$

$$\times \bar{R}^{\bar{V}_2 \cdots \bar{V}_{n-1}]}_{((t^{JK})' \omega^{\bar{V}_1 \cdots \bar{V}_n})},$$

$$R^{J_1 \cdots J_p}_{r^{\{J_p\}}} * \bar{U}^{\bar{V}_1 \cdots \bar{V}_m}_{\omega^{\{\bar{V}_m\}}}$$

$$= -\tfrac{1}{2} i(i)^{\{[\frac{p}{2}]-[\frac{p+2}{2}]\}}$$

$$\times \delta^{[\bar{V}_1 \cdots \bar{V}_{m-2}}_{[J_1 \cdots J_p]} \delta^{\bar{V}_{m-1}],\bar{V}_m} \delta^{m,p+2} \bar{L}_{(r\omega)''}$$

$$+ i^{\{[\frac{p}{2}]-[\frac{p+1}{2}]\}} \delta^{p+1,m} \bar{G}^{\bar{V}_m}_{(r\omega)'} \delta^{[\bar{V}_1 \cdots \bar{V}_{m-1}]}_{[J_1 \cdots J_p]}$$

$$+ 2i(-1)^{p+1} \delta^{p+3,m} (i)^{[\frac{p}{2}]-[\frac{p+2}{2}]}$$

$$\times \bar{G}^{[\bar{V}_1}_{(r\omega)''} \delta^{\bar{V}_2 \cdots \bar{V}_{m-2}}_{[J_1 \cdots J_p]} \delta^{\bar{V}_{m-1}],\bar{V}_m}$$

$$+ i(-1)^{p} \bar{T}^{\bar{V}_1[\bar{V}_m}_{(r\omega)'} \delta^{\bar{V}_2 \cdots \bar{V}_{m-1}]}_{[J_1 \cdots J_p]} \delta^{p+2,m}$$

$$+ (-1)^{pm} (i)^{\{[\frac{m}{2}]+[\frac{p}{2}]-[\frac{m+p-2}{2}]\}}$$

$$\times \bar{U}^{\bar{V}_{p+1} \cdots \bar{V}_{m-p}}_{(r\omega)'} \delta^{\bar{V}_1 \cdots \bar{V}_p}_{[J_1 \cdots J_p]},$$

$$U^{I_1 \cdots I_q}_{\mu^{\{I_q\}}} * \bar{U}^{\bar{V}_1 \cdots \bar{V}_m}_{\omega^{\{\bar{V}_m\}}}$$

$$= -\delta^{m\,q} \delta^{[I_1 \cdots I_q]}_{[\bar{V}_1 \cdots \bar{V}_q]} \bar{L}_{((\frac{4-q}{2})\mu\omega - (\frac{q-2}{2})\mu\omega')}$$

$$- 2\bar{T}^{\bar{V}_m J_q}_{(\omega\mu)} \delta^{[I_1 \cdots I_q]}_{[\bar{V}_1 \cdots \bar{V}_q]} \delta^{q}_{m}$$

$$- 2i^{([\frac{\bar{V}_1}{2}]-[\frac{\bar{V}_1+1}{2}])} \delta^{m,(q+1)}$$

$$\times \bar{G}^{[\bar{V}_1}_{(-(q-2)\omega'\mu + (3-q)\omega\mu')} \delta^{\bar{V}_2 \cdots \bar{V}_m]}_{[I_1 \cdots I_q]}$$

$$+ 2(-1)^{q}(i)^{[\frac{q}{2}]-[\frac{q+1}{2}]} \bar{G}^{I_q}_{(-\omega\mu)'} \delta^{m,q+1}$$

$$\times \delta^{[\bar{V}_1 \cdots \bar{V}_{m-2}}_{[I_1 \cdots I_{q-1}]} \delta^{\bar{V}_{m-1}],\bar{V}_m}$$

$$- i(i)^{[\frac{q}{2}]-[\frac{q-1}{2}]} \sum_{r=1}^{q-1} (-1)^{r-1} \delta^{m}_{q-1}$$

$$\times \bar{G}^{[I_r}_{(\omega\mu)} \delta^{I_1}_{[\bar{V}_1} \cdots \delta^{I_{r-1}}_{\bar{V}_{r-1}} \delta^{I_{r+1}}_{\bar{V}_r} \cdots \delta^{I_q]}_{\bar{V}_m]}$$

$$+ \sum_{r=1}^{q-1} 2(-1)^{r+1}$$

$$\times \left(\bar{T}^{J_r[\bar{V}_1}_{(\omega\mu)} \delta^{\bar{V}_2}_{[I_1} \cdots \delta^{\bar{V}_r}_{I_{r-1}} \delta^{\bar{V}_{r+1}}_{I_{r+1}} \cdots \delta^{]\bar{V}_m}_{I_q]} \delta^{q,m} \right)$$

$$+ i(i)^{\{[\frac{q}{2}]+[\frac{m-q}{2}+2]-[\frac{m+2}{2}]\}}$$

$$\times \left\{ \sum_{r=1}^{q} (-1)^{r-1} \delta^{[\bar{V}_1 \cdots \bar{V}_{q-1}}_{[I_1 \cdots I_{q-1}]} \right.$$

$$\times \overline{U}^{\bar{V}_q \cdots \bar{V}_{q+r-1} I_q \bar{V}_{q+r} \cdots]V_m}_{\omega\mu}$$

$$-(-1)^{q(m-q+2)} \sum_{r=1}^{q} (-1)^{r-1}$$

$$\overline{U}^{\bar{V}_1 \cdots \bar{V}_{m-q+1}[I_r}_{\omega\mu} \delta^{I_1 \cdots I_{r-1} I_{r+1} \cdots I_q]}_{\bar{V}_{m-q+2} \cdots \bar{V}_{m-q+2+r} \cdots \bar{V}_m} \Bigg\}$$

$$-(i)^{\{[\frac{q}{2}]+[\frac{m-q}{2}]-[\frac{q+m-4}{2}]\}}$$

$$\times \overline{R}^{[\bar{V}_1 \cdots \bar{V}_{m-q}}_{(\omega\mu)'} \delta^{\bar{V}_{m-q+1} \cdots]\bar{V}_m}_{[I_1 \cdots]I_q} ,$$

$$R^{J_1 \cdots J_p}_{r^{\{J_p\}}} * \overline{R}^{\bar{T}_1 \cdots \bar{T}_m}_{\rho^{\{T_m\}}}$$

$$= \delta^{[\bar{T}_1 \cdots \bar{T}_m]}_{[J_1 \cdots J_p]} \delta^{p\,m} \overline{L}_{(-(\frac{p}{2}-2)r'\rho-(\frac{p}{2}-1)r\rho')} + (-1)^p$$

$$\times \Bigg\{ 2(i)\overline{G}^{[\bar{T}_1}_{((2-p)r'\rho-(p-1)r\rho')} \delta^{\bar{T}_2 \cdots \bar{T}_m]}_{[J_1 \cdots J_p]} \delta^m_{p+1}$$

$$+(i)(i)^{[\frac{p}{2}]-[\frac{p-1}{2}]}$$

$$\times \sum_{r=1}^{p} (-1)^r \overline{G}^{J_r}_{(r\rho)} \delta^m_{p-1} \delta^{[J_1 \cdots J_{r-1} J_{r+1} \cdots J_p]}_{[\bar{T}_1 \cdots \bar{T}_{r-1} \bar{T}_r \cdots \bar{T}_m]} \Bigg\}$$

$$+ \sum_{r=1}^{p} (-1)^{r+1} 2\overline{T}^{[\bar{T}_1 | J_r|}_{(r\rho)} \delta^{\bar{T}_2 \cdots \bar{T}_m]}_{[J_1 \cdots J_{r-1} J_{r+1} \cdots J_p]} \delta^p_m$$

$$- \sum_{r=1}^{p} (-1)^{r-1}$$

$$\times \overline{U}^{[\bar{T}_1 \cdots \bar{T}_{m-p+1} | J_r|}_{(r\rho)} \delta^{\bar{T}_{m-p+2} \cdots \bar{T}_m]}_{J_1 \cdots J_{r-1} J_{r+1} \cdots J_p}$$

$$+ i^{\{[\frac{p}{2}]+[\frac{m-p}{2}]-[\frac{m}{2}]\}} \overline{R}^{[\bar{T}_{p+1} \cdots \bar{T}_m}_{(2r'\rho+r\rho')} \delta^{\bar{T}_1 \cdots \bar{T}_p]}_{[J_1 \cdots J_p]} ,$$

$$T^{JK}_{t^{JK}} * \overline{R}^{\bar{T}_1 \cdots \bar{T}_m}_{\rho^{\bar{T}_1 \cdots \bar{T}_m}}$$

$$= \sum_{r=1}^{m} (-1)^{r+1} \Big(\delta^{[\bar{T}_1 | J|} \overline{R}^{\bar{T}_2 \cdots \bar{T}_{r-1} \underline{\perp} K | \bar{T}_{r+1} \cdots \bar{T}_m]}_{(t^{JK} \rho^{\bar{V}_1 \cdots \bar{V}_n)}}$$

$$- \delta^{[\bar{T}_1 | K|} \overline{R}^{\bar{T}_2 \cdots \bar{T}_{r-1} \underline{\perp} J | \bar{T}_{r+1} \cdots \bar{T}_m]}_{(t^{JK} \rho^{\bar{V}_1 \cdots \bar{V}_n)}} \Big) ,$$

$$U^{I_1 \cdots I_q}_{\mu^{\{I_q\}}} * \overline{R}^{\bar{T}_1 \cdots \bar{T}_m}_{\rho^{\{T_m\}}}$$

$$= -i(-1)^{q(m-q+2)}(i)^{\{[\frac{m-q}{2}+2]+[\frac{q}{2}]-[\frac{m}{2}]\}}$$

$$\times \sum_{r=1}^{m-q+2} (-1)^{r-1} \delta^{[I_1 \cdots I_{q-1}]}_{[T_1 \cdots T_{q-1}]}$$

$$\times \overline{R}^{\bar{T}_1 \cdots \bar{T}_{q+r-1} I_q \bar{T}_{q+r+1} \cdots \bar{T}_m}_{\rho\mu} ,$$

$$T^{JK}_{t^{JK}} * \Big(\overline{T}^{\bar{R}\bar{S}}_{\tau^{\bar{R}\bar{S}}}, \overline{\beta} \Big)$$

$$= \tfrac{1}{2}\big(\delta^{\bar{R}J}\delta^{\bar{S}K} - \delta^{\bar{R}K}\delta^{\bar{S}J} \big) \overline{L}_{((t^{JK})'\,\tau^{\bar{R}\bar{S}})}$$

$$+ \tfrac{1}{2} \overline{T}^{A\,B}_{(t^{JK}\,\tau^{\bar{R}\bar{S}})} \delta^{JK\bar{R}\bar{S}}_{AB} + 4\overline{\beta}\,\overline{T}^{JK}_{(\tau^{\bar{R}\bar{S}})'} ,$$

where

$$\delta^{JK\bar{R}\bar{S}}_{AB}$$

$$\equiv \big(\delta^{AK}\delta^{B\bar{S}}\delta^{\bar{R}J} - \delta^{AK}\delta^{B\bar{R}}\delta^{\bar{S}J} + \delta^{A\bar{S}}\delta^{BJ}\delta^{\bar{R}K}$$

$$- \delta^{A\bar{R}}\delta^{JS}\delta^{\bar{S}K} + \delta^{A\bar{S}}\delta^{BK}\delta^{\bar{R}J} - \delta^{A\bar{R}}\delta^{KB}\delta^{\bar{S}J}$$

$$+ \delta^{AJ}\delta^{B\bar{S}}\delta^{\bar{R}K} - \delta^{JA}\delta^{\bar{R}B}\delta^{\bar{S}K} + \delta^{A\bar{S}}\delta^{BK}\delta^{\bar{R}J} \big) , \tag{39}$$

where the symmetry of the indices on the left hand side should be imposed on the indices on the right side. In the above, we have sometimes suppressed the indices associated with the functions used by the generators. For example $\omega^{\bar{V}_1 \cdots \bar{V}_n}$, the associate of the $\overline{U}$ dual element, may be written as $\omega^{\{V_m\}}$ or simply as ω. Also the notation $\delta^{I_1 \cdots I_m}_{J_1 \cdots J_m} \equiv \delta^{I_1}_{J_1} \cdots \delta^{I_m}_{J_m}$ was utilized.

There is quite a bit of interchange between functions in various sectors suggesting that the inhomogeneous contribution in the transformation laws for D, is the interaction of the central extension with D. The coadjoint of the Virasoro algebra, i.e. the action of L on the coadjoint vectors reveals a spectrum of states containing:

- D corresponds to a rank 2 covariant tensor when the central extension is set to zero and is otherwise a quadratic differential.
- $\psi^{\bar{I}}$ corresponds to N spin-$\frac{3}{2}$ fields that partner with D.
- $\tau^{\bar{R}\bar{S}}$ corresponds to the spin-1 covariant tensors that serves as the $N(N-1)/2$ SO(N) gauge potentials associated with the supersymmetries.
- Given the N supersymmetries, there are the fields $\omega^{\bar{V}_1 \cdots \bar{V}_p}$. For a fixed value of N, the total number of independent components is given by

$$\#(U) = N(2^N - N - 1) .$$

- Again given N supersymmetries, there are the fields $\rho^{\bar{T}_1 \cdots \bar{T}_p}$. For a fixed value of N, the total number of independent components is given by

$$\#(R) = (2^N - N - 1) .$$

The spins of the fields associated with U and R vary according to $(2-\frac{p}{2})$. These likely correspond to other gauge and non-gauge physical fields, auxiliary, and Stuckelberg fields that are required to close the supersymmetry algebra. For a fixed level number $\mathscr{A}$, there are $(N+1)2^N - \frac{1}{2}(N+3)N$ independent vector fields for $N > 4$.

We end our discussion with a conjecture. If M-theory possesses a 1D NSR formulation, it seems likely that the $N = 32$ or 16 case of the present discussion determines the structure of its representation. We conjecture that the spectrum of the 1D, $N = 32$ or 16 $\mathscr{G}R$ super Virasoro theory provides a set of fundamental NSR variables to describe M-theory.

''*Equations were drawn up in paisley form.*'' – Rakim (1997)

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