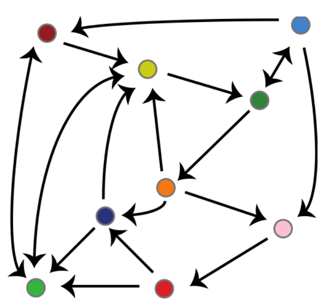
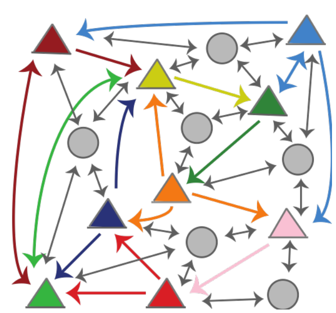
So far, everything we have done for CTLNs/gCTLNs has assumed negative (inhibitory) weights on the W matrix.

graph G



Idea: network of excitatory and inhibitory cells



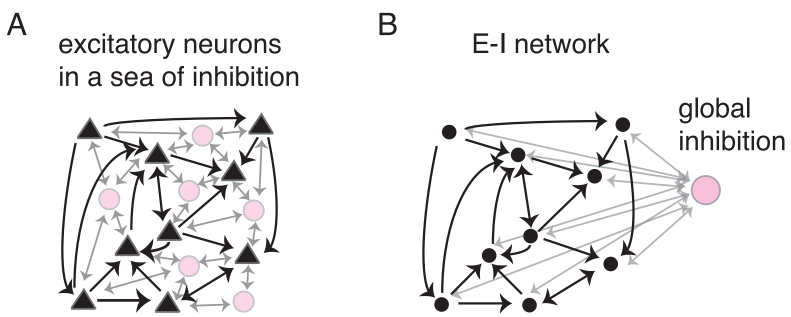
The gCTLN is defined by a graph G and two vectors of parameters:

$$W_{ij} = \begin{cases} -1+\varepsilon_j & \text{if } j \to i, \\ -1-\delta_j & \text{if } j \not\to i, \\ 0 & \text{if } i = j. \end{cases}$$

weak inhibition

strong inhibition

E-I TLNs from graphs



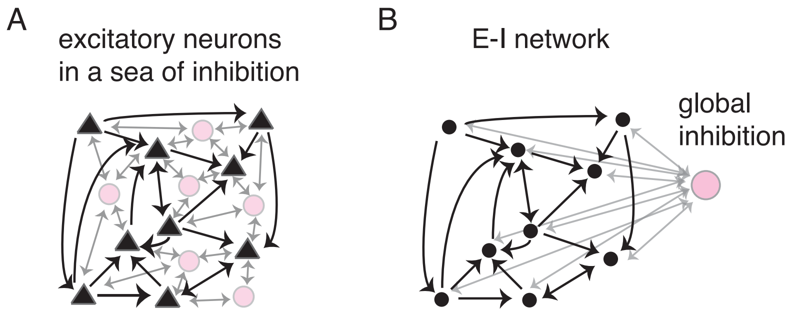
$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + W_{iI}(x_I - W_{Ii}x_i) + b_i\right]_+, \; i = 1, \ldots, n,$$

$$\frac{dx_I}{dt} = \frac{1}{\tau_I}\left(-x_I + \left[\sum_{j=1}^{n} W_{Ij}x_j + b_I\right]_+\right).$$

$$W_{ij} = \begin{cases} a_j & \text{if } j \to i \text{ in } G, \\ 0 & \text{if } j \not\to i \text{ in } G, \\ 0 & \text{if } i = j, \end{cases} \quad \text{and} \quad \begin{aligned} W_{Ij} &= c_j, \\ W_{iI} &= -1, \\ W_{II} &= 0. \end{aligned}$$

Curto 2025 (preprint soon!)

E-I TLNs from graphs



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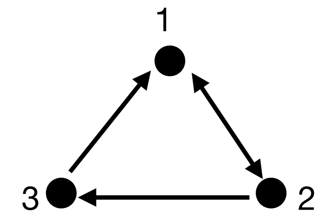
Example G:



W for E-I TLN

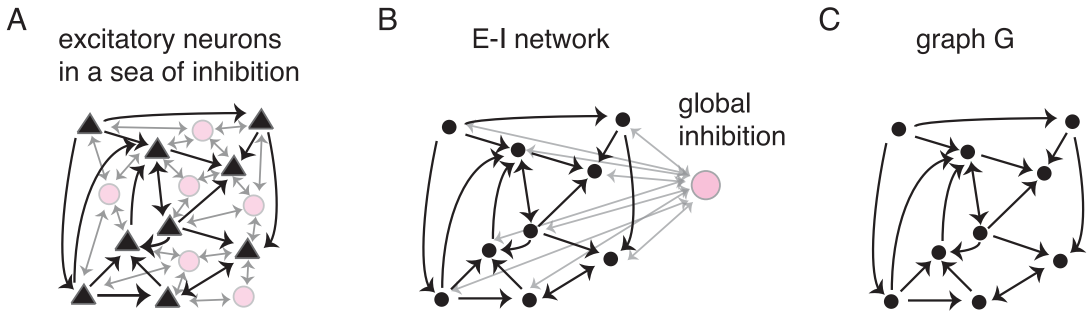
$$W = \begin{pmatrix} 0 & a_2 & a_3 & -1 \\ a_1 & 0 & 0 & -1 \\ 0 & a_2 & 0 & -1 \\ c_1 & c_2 & c_3 & 0 \end{pmatrix}$$

W for gCTLN

$$W = \begin{pmatrix} 0 & -1+\varepsilon_2 & -1+\varepsilon_3 \\ -1+\varepsilon_1 & 0 & -1-\delta_3 \\ -1-\delta_1 & -1+\varepsilon_2 & 0 \end{pmatrix}$$

Curto 2025 (preprint soon!)

There is a mapping from E-I TLNs to gCTLNs that preserves fixed points



inhibitory interaction

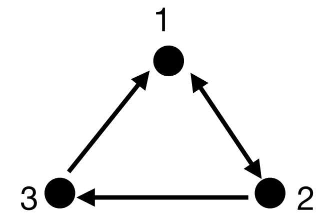
$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \boxed{W_{iI}(x_I - W_{Ii}x_i)} + b_i\right]_+, \quad i = 1, \ldots, n,$$

$$\frac{dx_I}{dt} = \frac{1}{\tau_I}\left(-x_I + \left[\sum_{j=1}^{n} W_{Ij}x_j + b_I\right]_+\right).$$

Parameter mapping to get the same fixed points:

$$\varepsilon_j = 1 + a_j - c_j,$$
$$\delta_j = c_j - 1.$$

Example G:



W for E-I TLN

$$W = \begin{pmatrix} 0 & a_2 & a_3 & -1 \\ a_1 & 0 & 0 & -1 \\ 0 & a_2 & 0 & -1 \\ c_1 & c_2 & c_3 & 0 \end{pmatrix}$$

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C. Lienkaemper, G. Ocker. Dynamics of clustered spiking networks via the CTLN model (2025)

Curto 2025 (preprint soon!)

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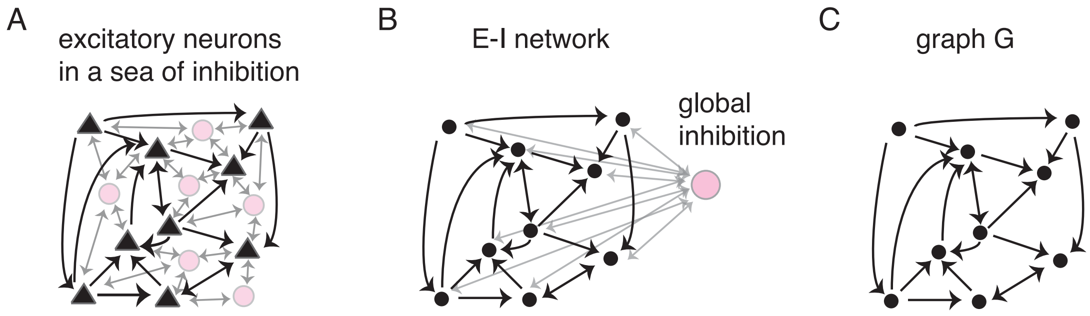

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Parameter mapping to get the same fixed points:

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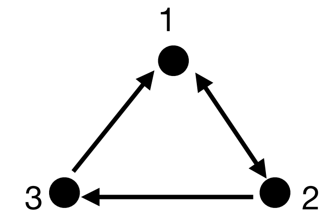
The mapping says nothing about the timescale of inhibition!

Example G:



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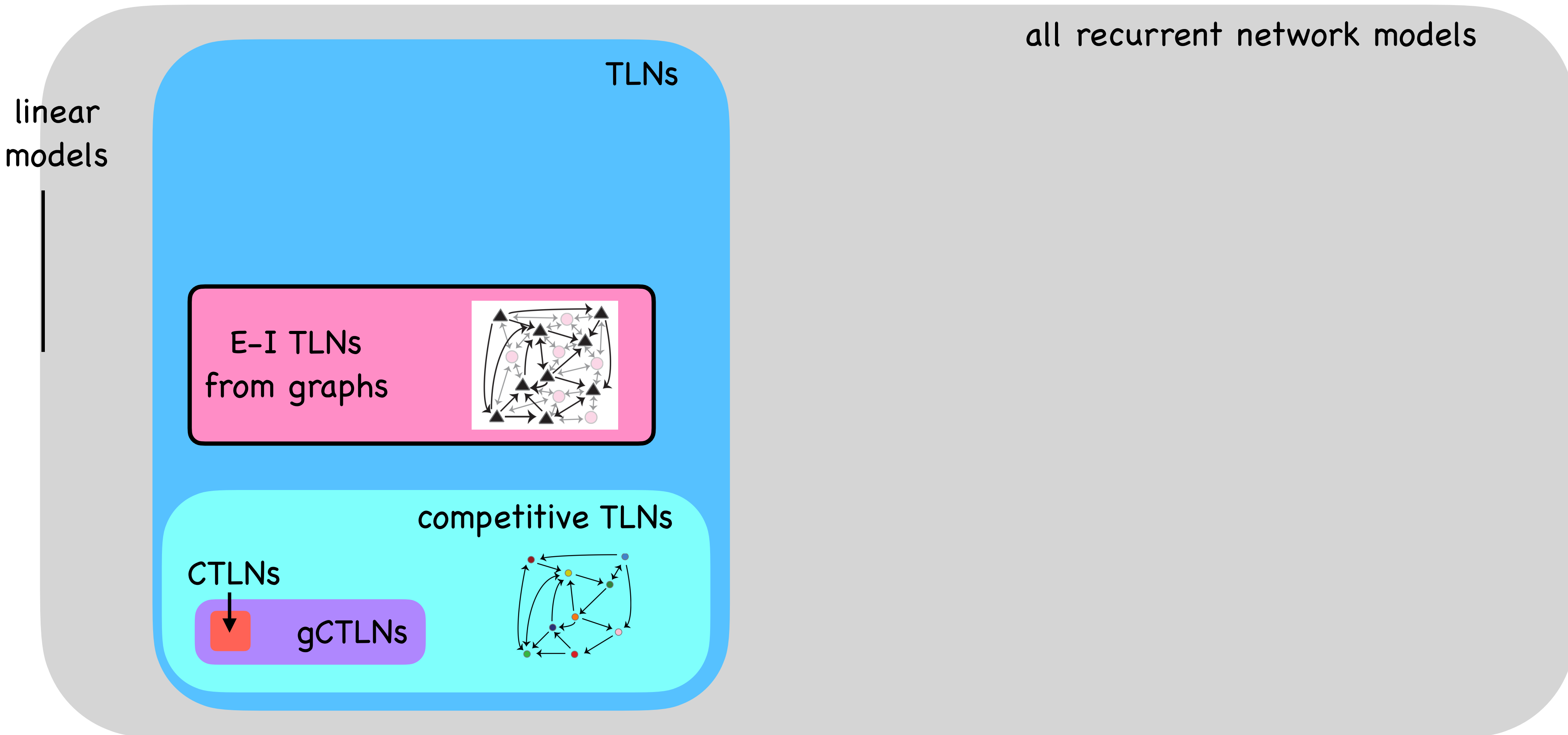
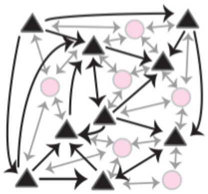
TLNs, CTLNs, and gCTLNs … and E-I TLNs from graphs

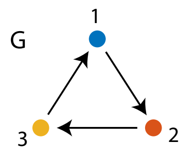


Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

excitatory neurons
in a sea of inhibition



A



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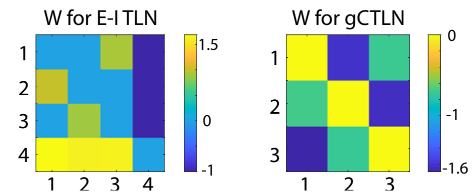

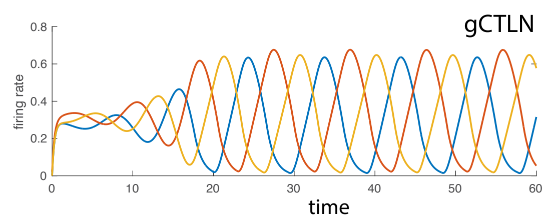

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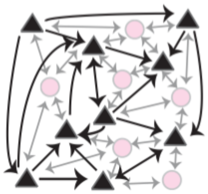

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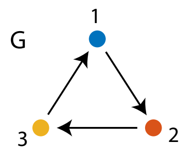

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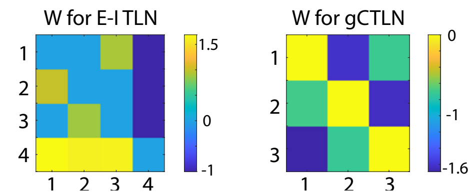

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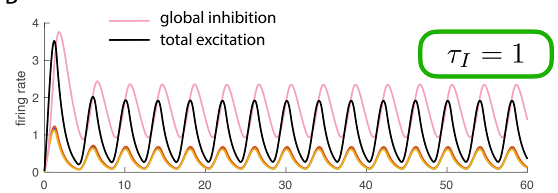

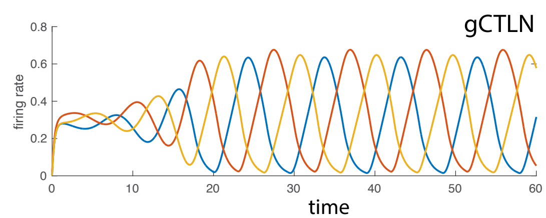

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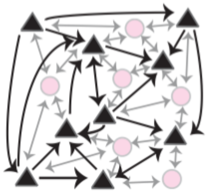

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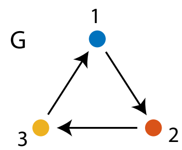

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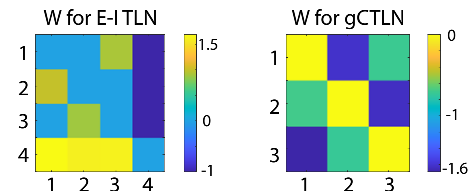

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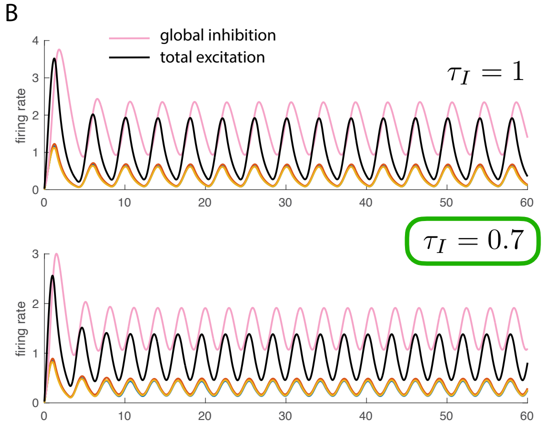
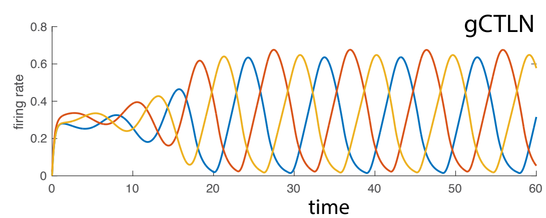
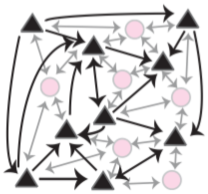




Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

excitatory neurons in a sea of inhibition



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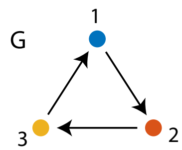

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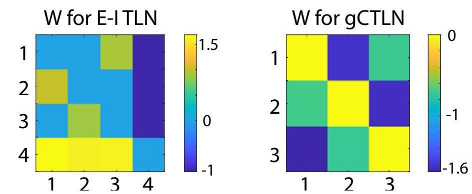

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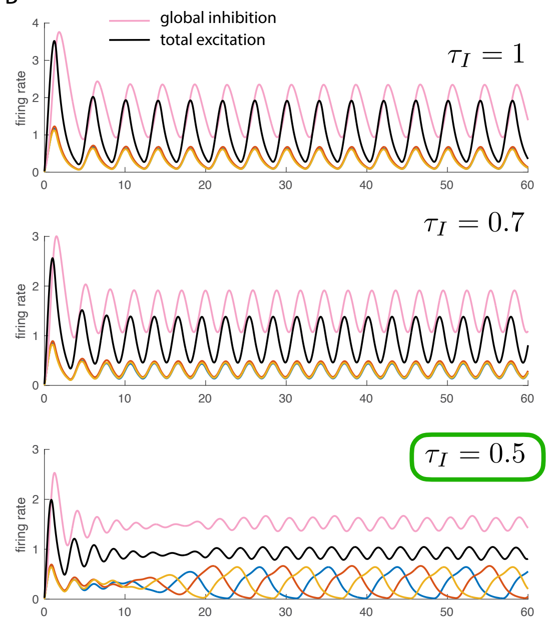
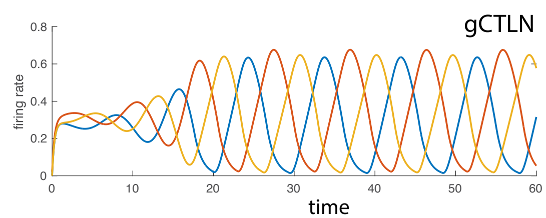
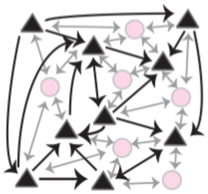
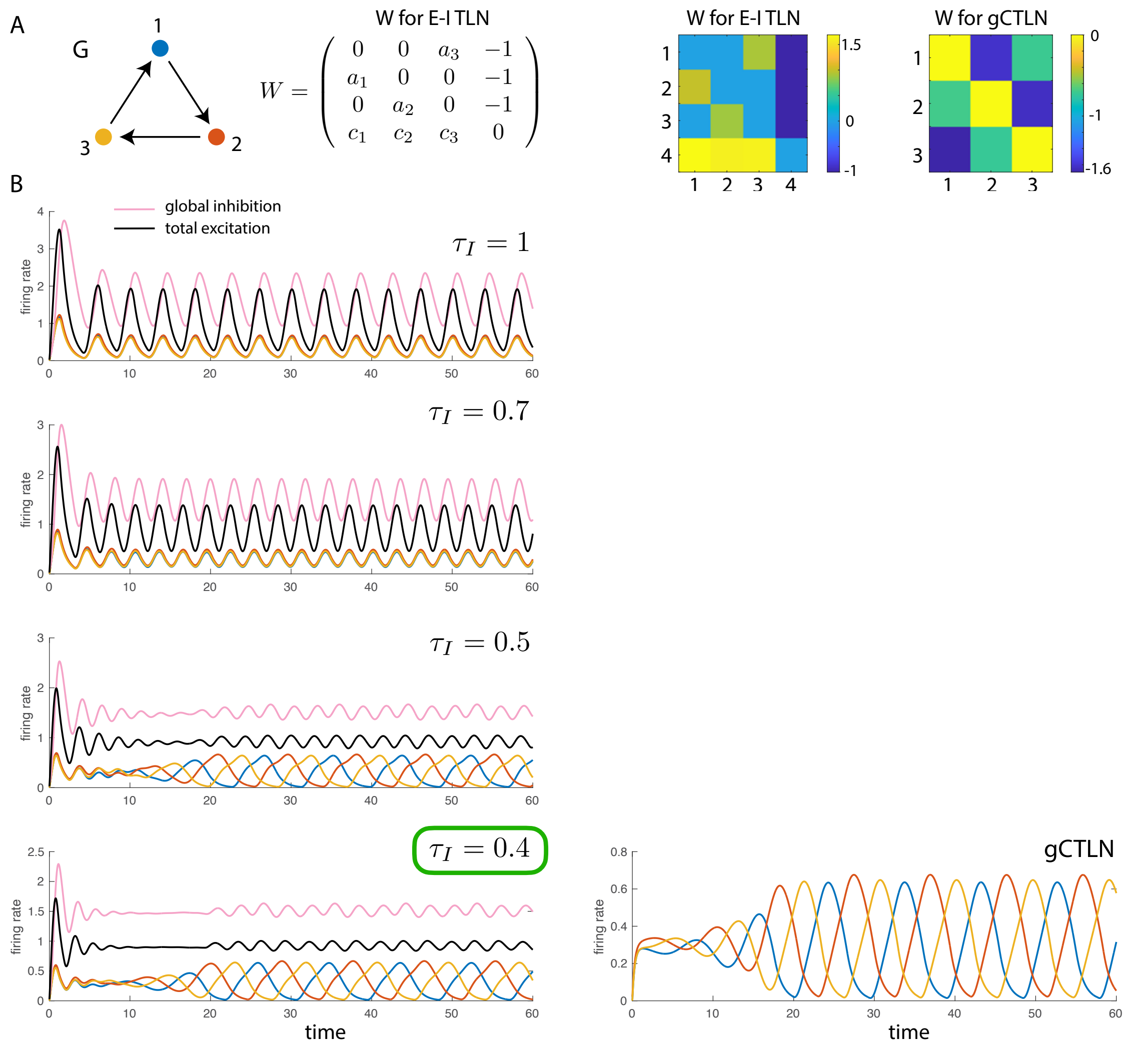




Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

excitatory neurons
in a sea of inhibition





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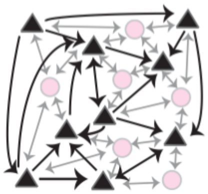

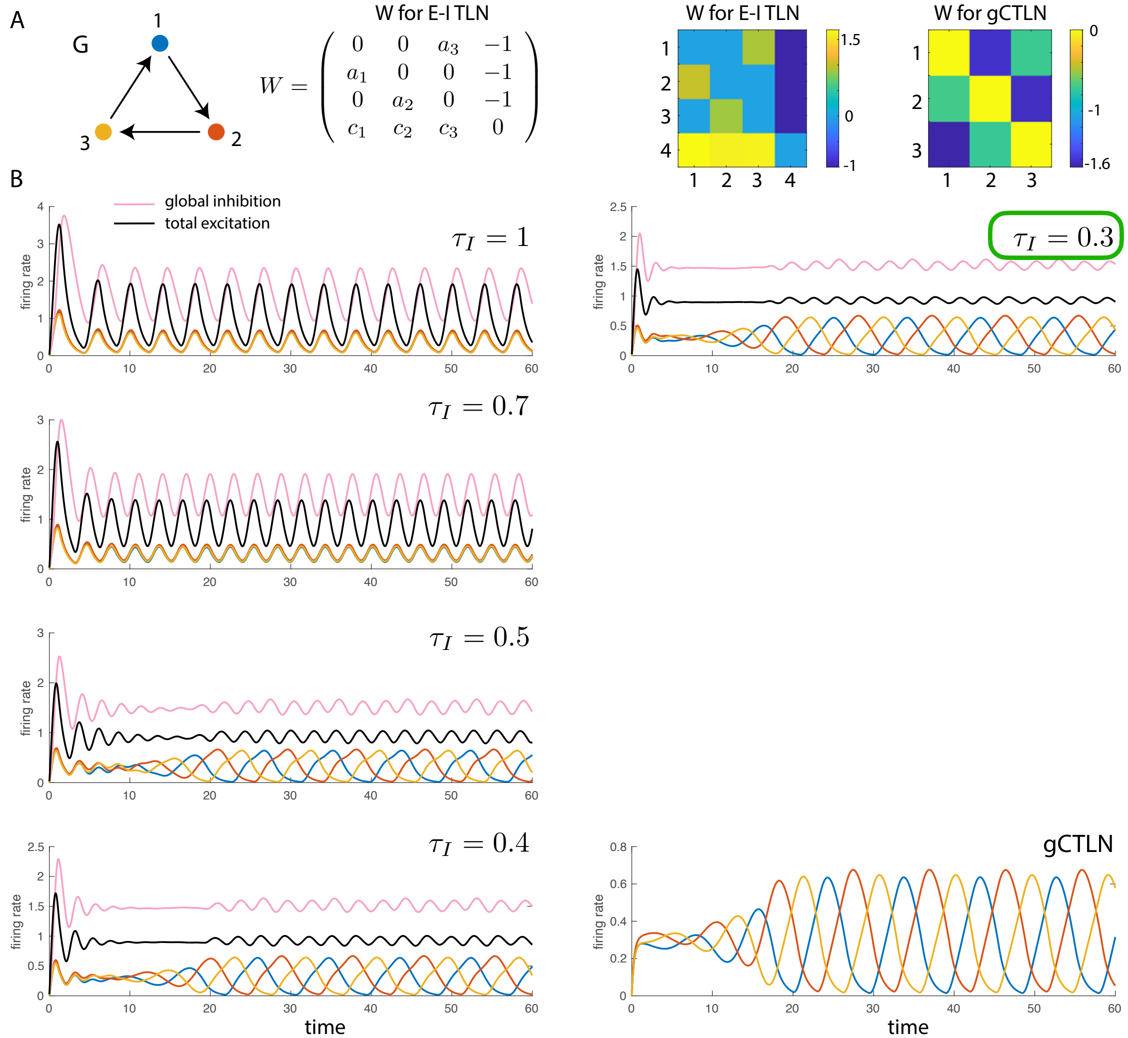

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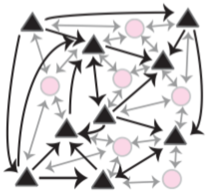
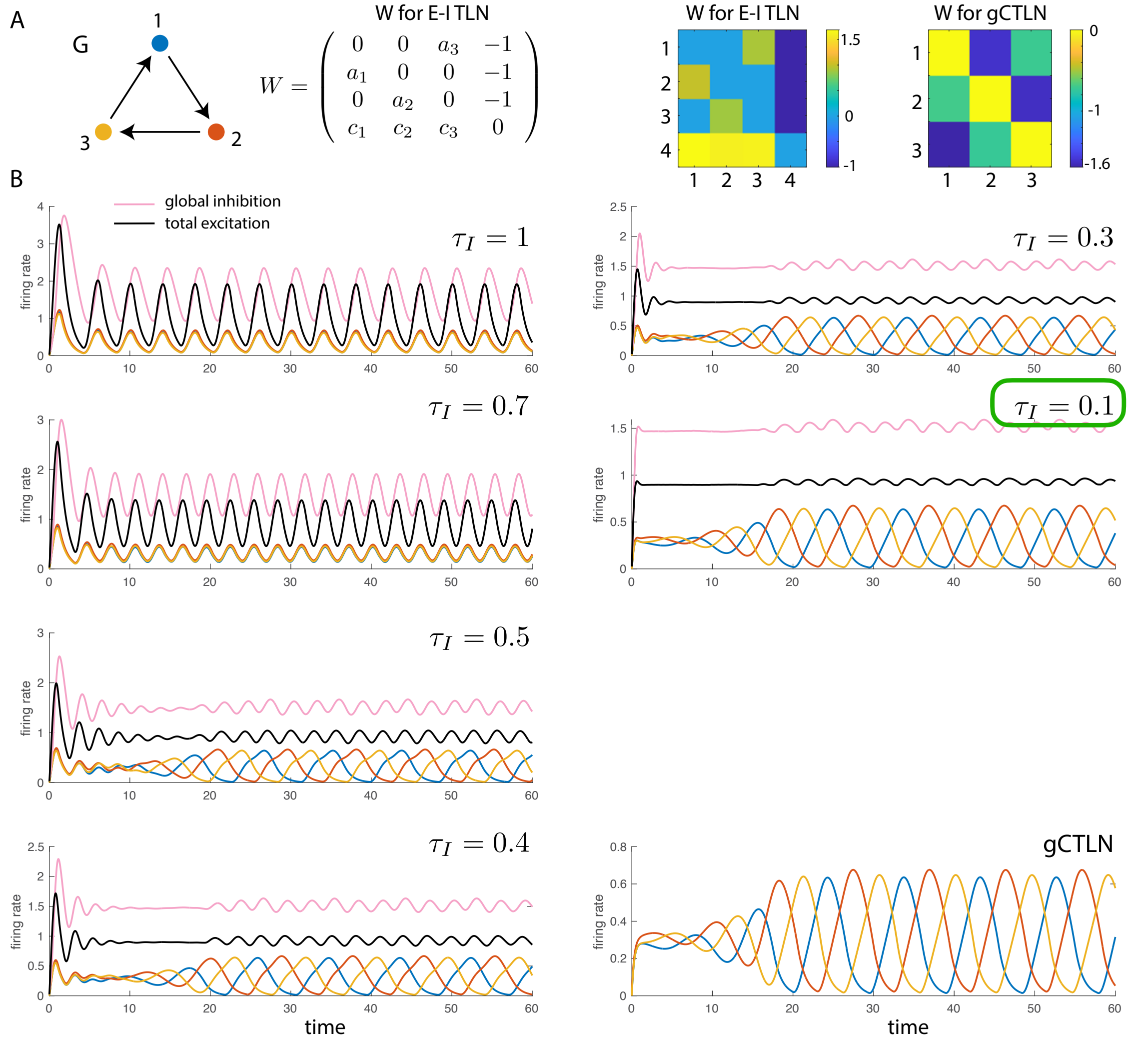
excitatory neurons in a sea of inhibition





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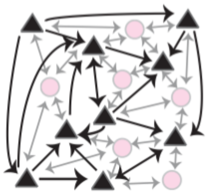
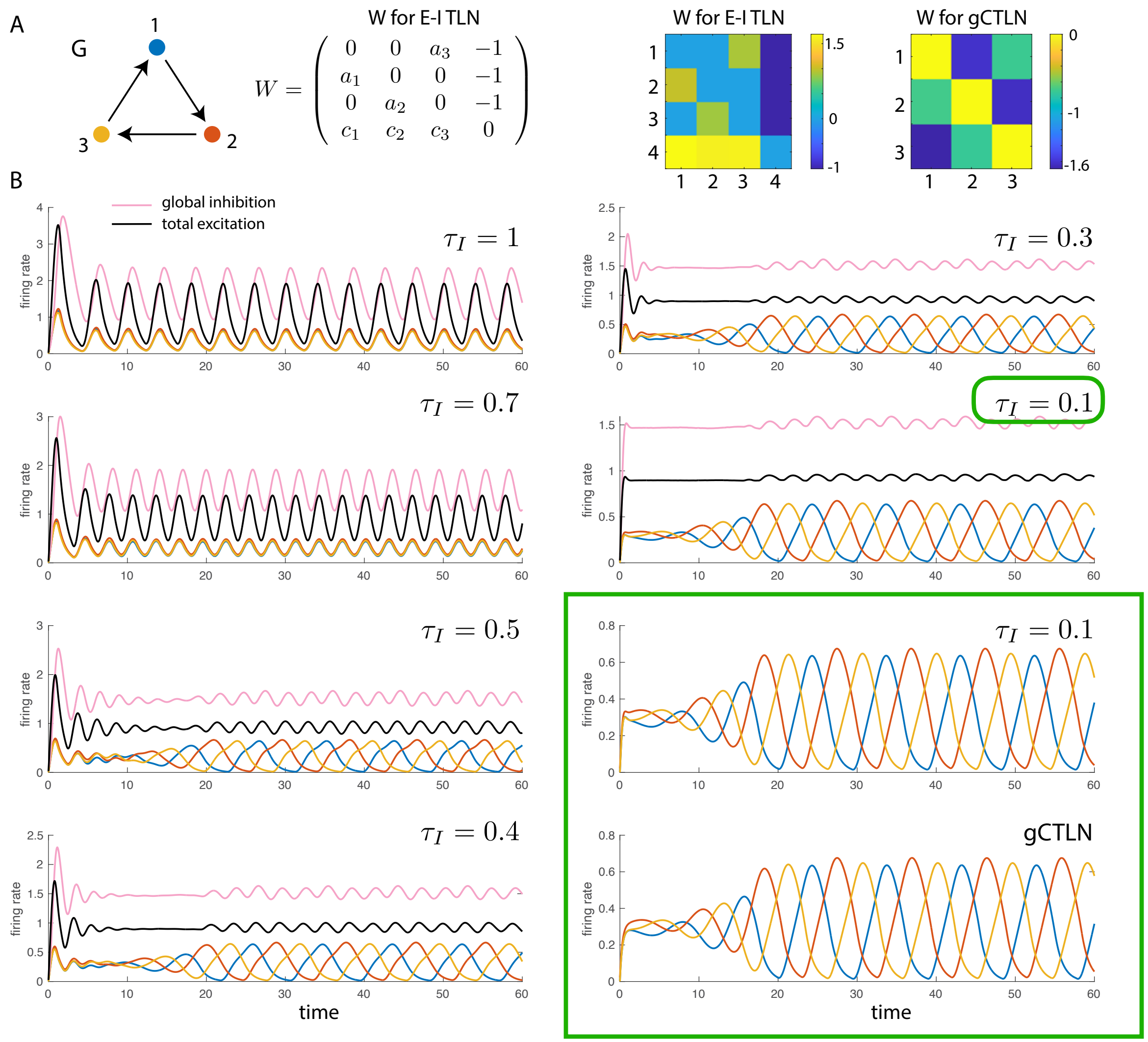
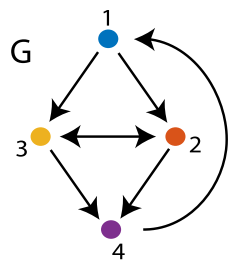




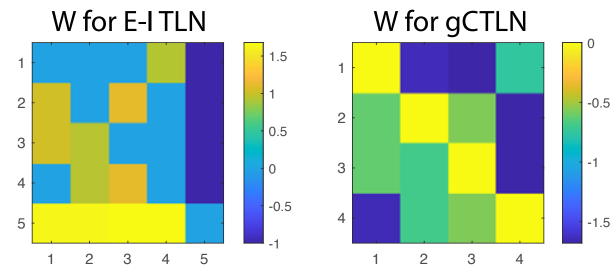
Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

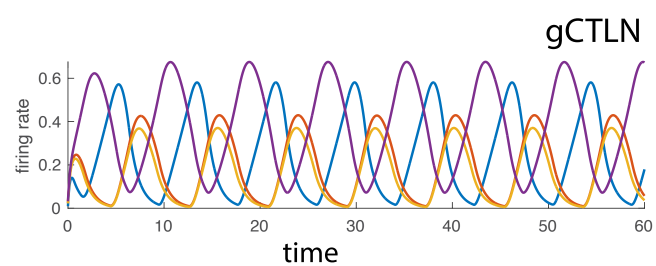
A



W for E-I TLN

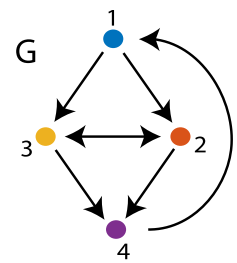
$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & -1 \\ a_1 & 0 & a_3 & 0 & -1 \\ a_1 & a_2 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{pmatrix}$$





Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

A



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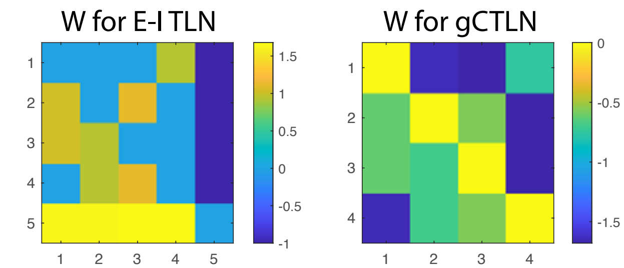
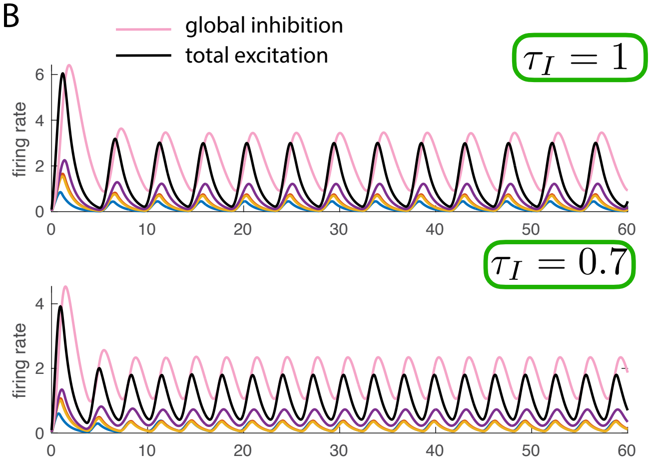
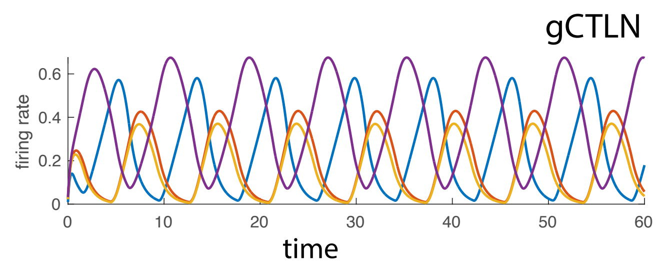
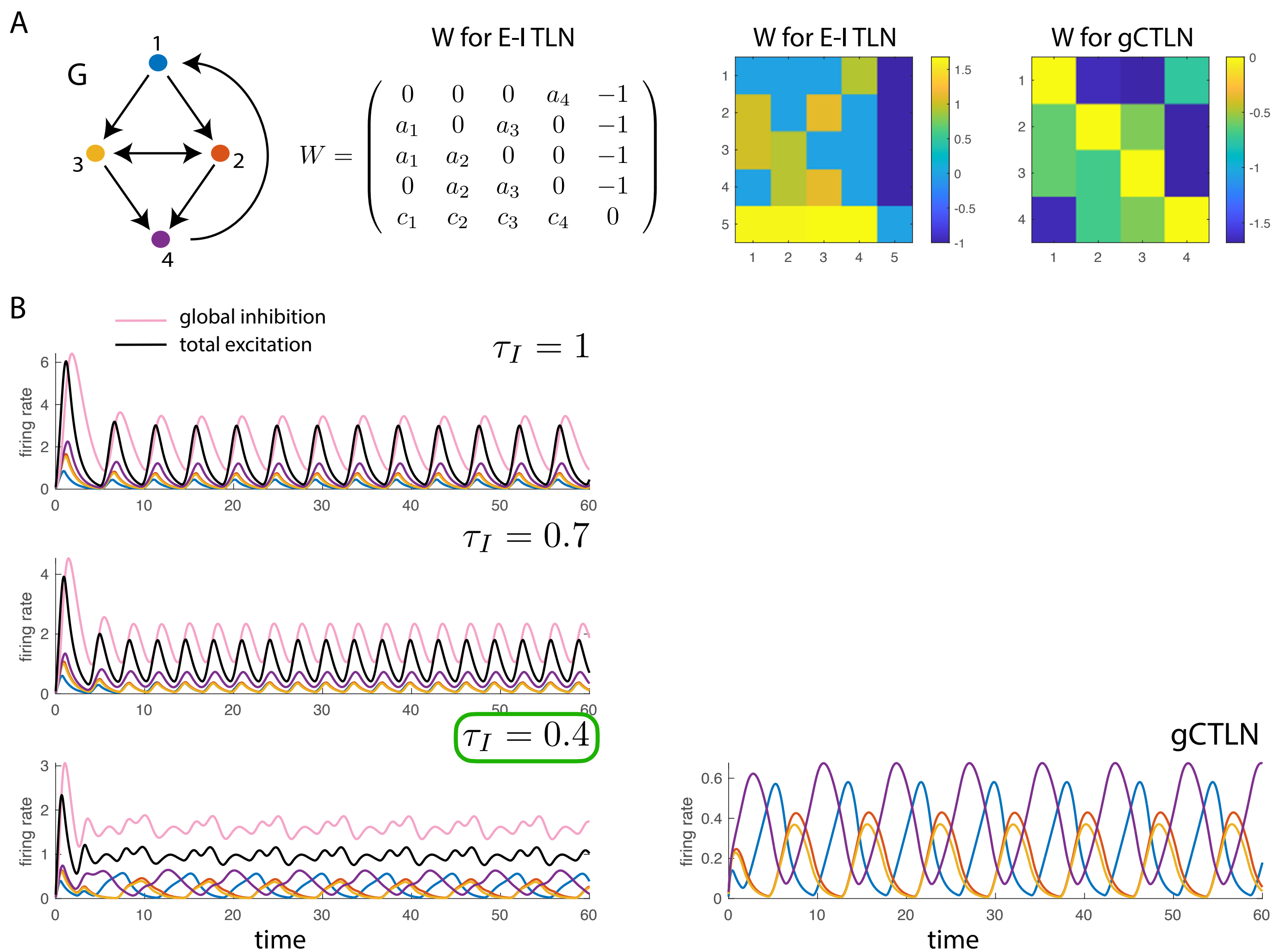


B





Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?



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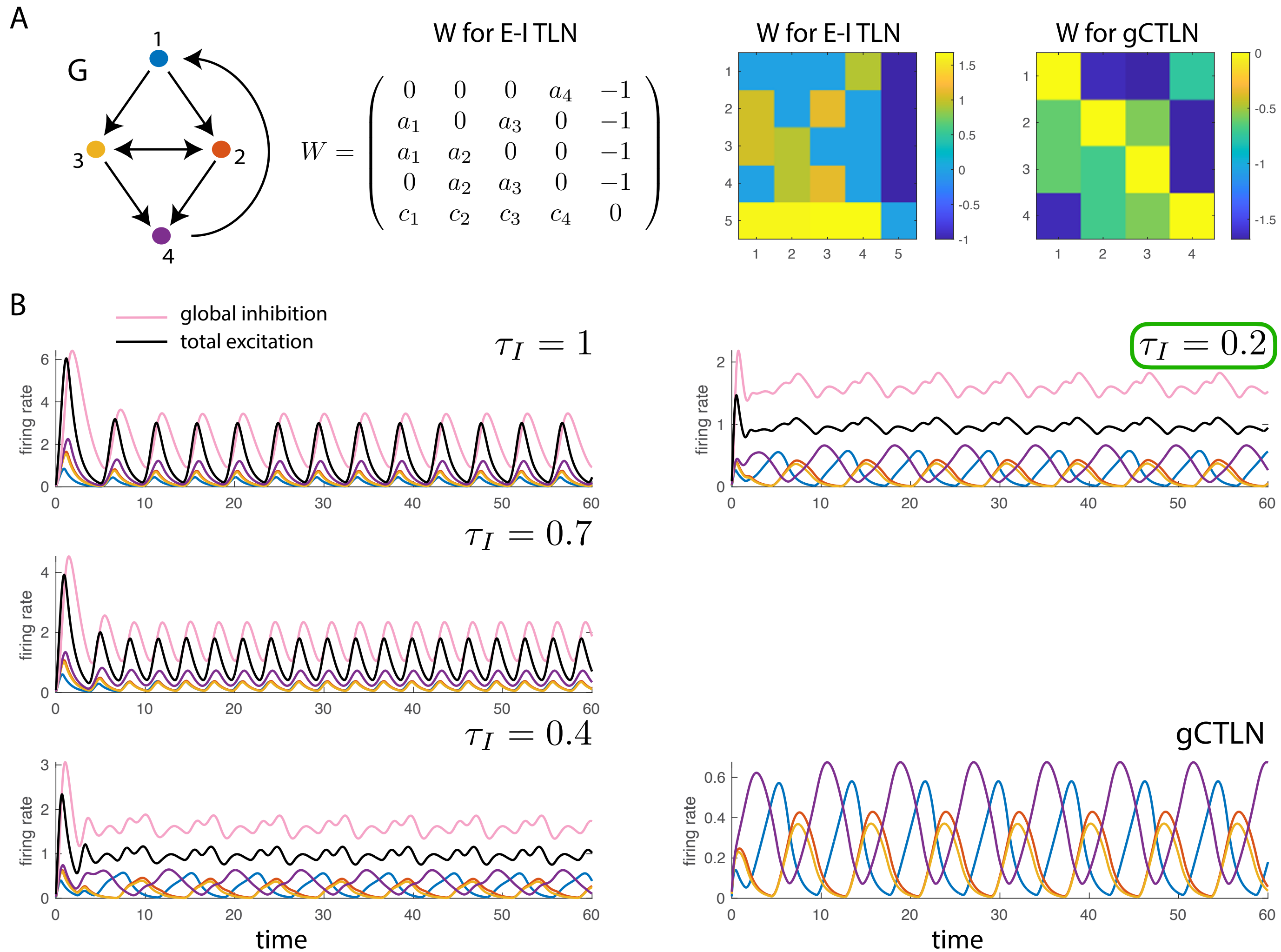

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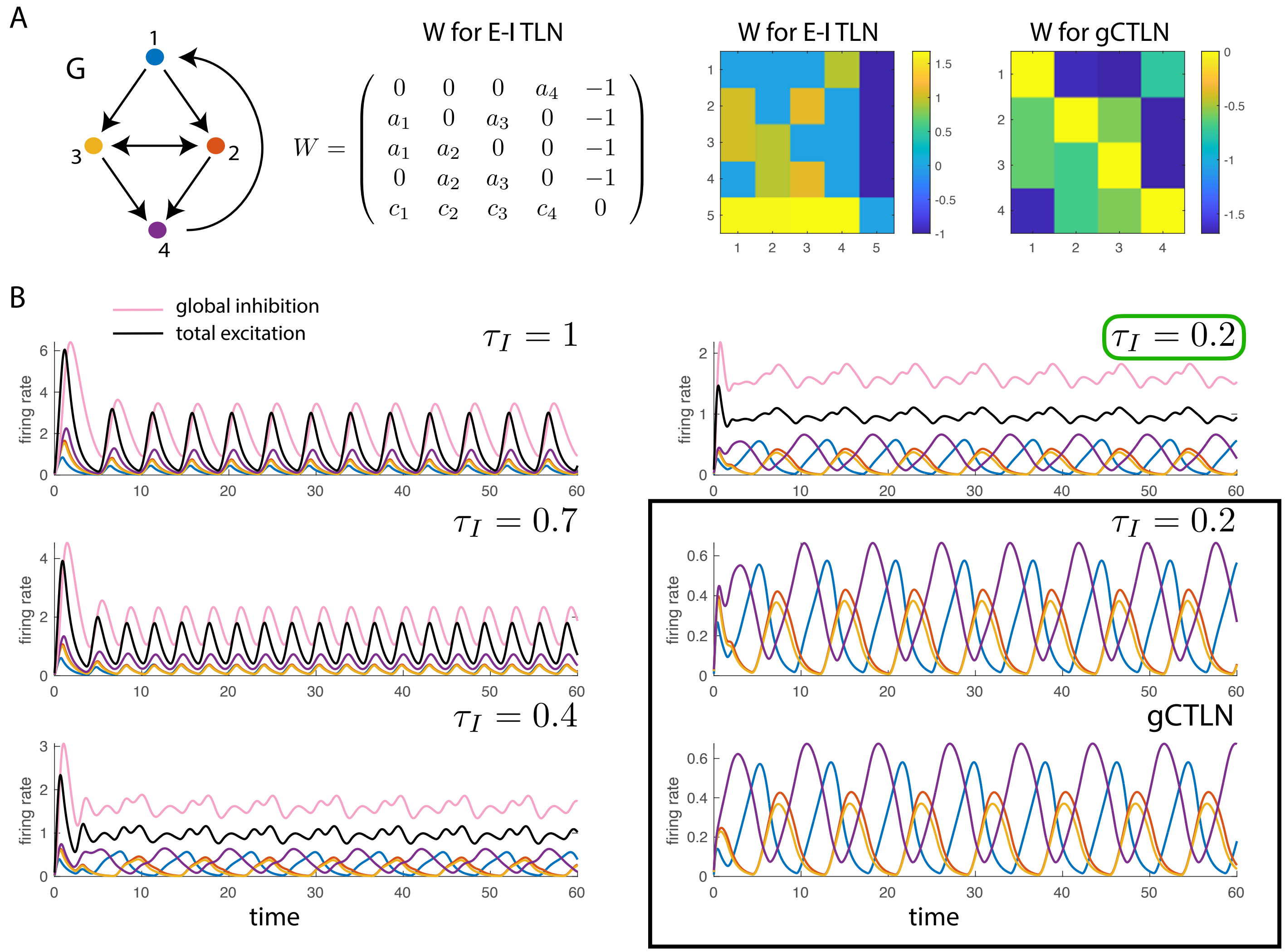
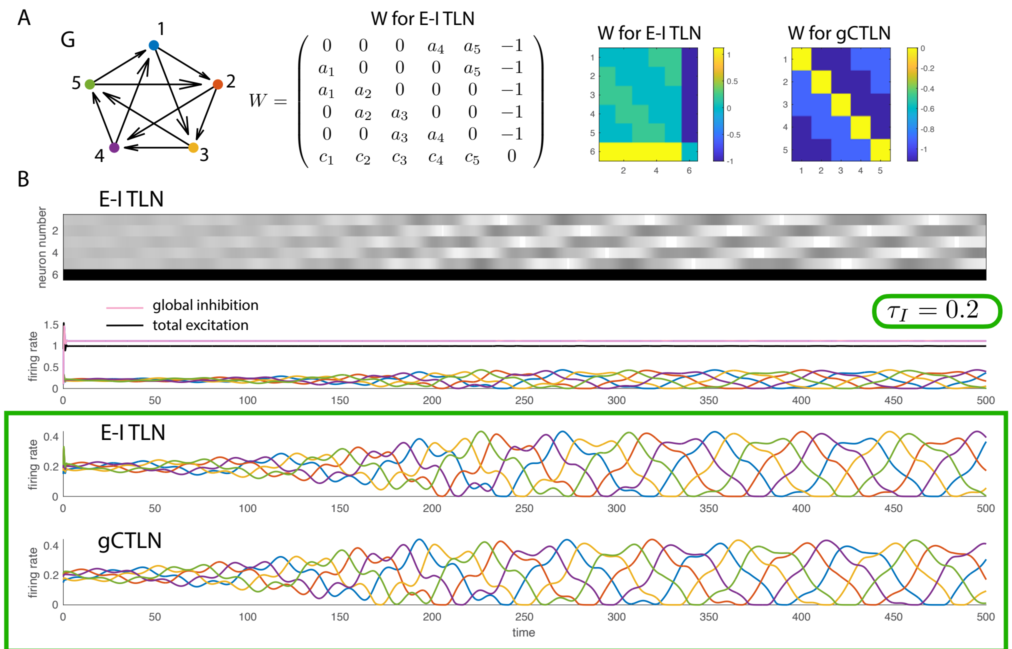
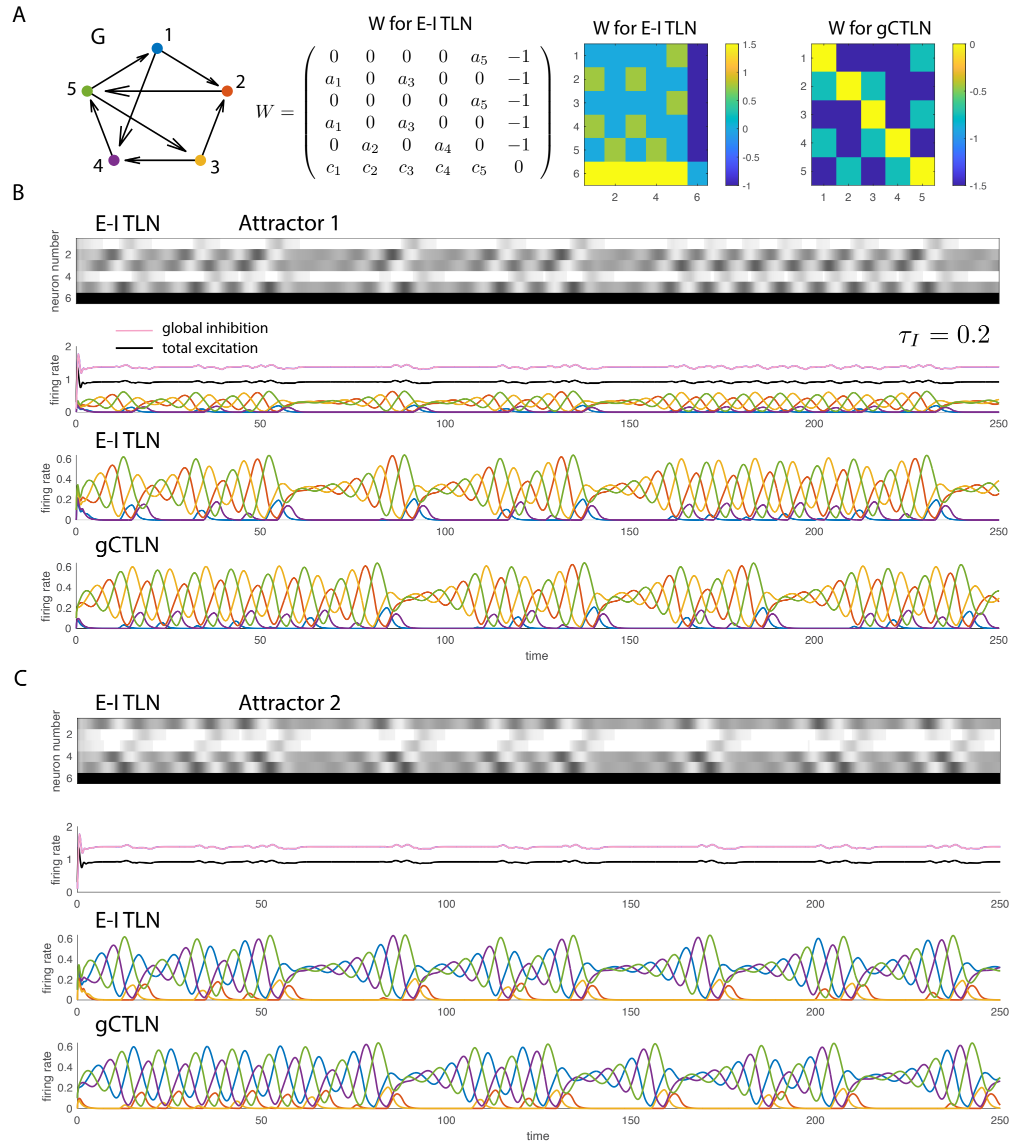


Even "exotic" attractors like Gaudi and baby chaos look the same





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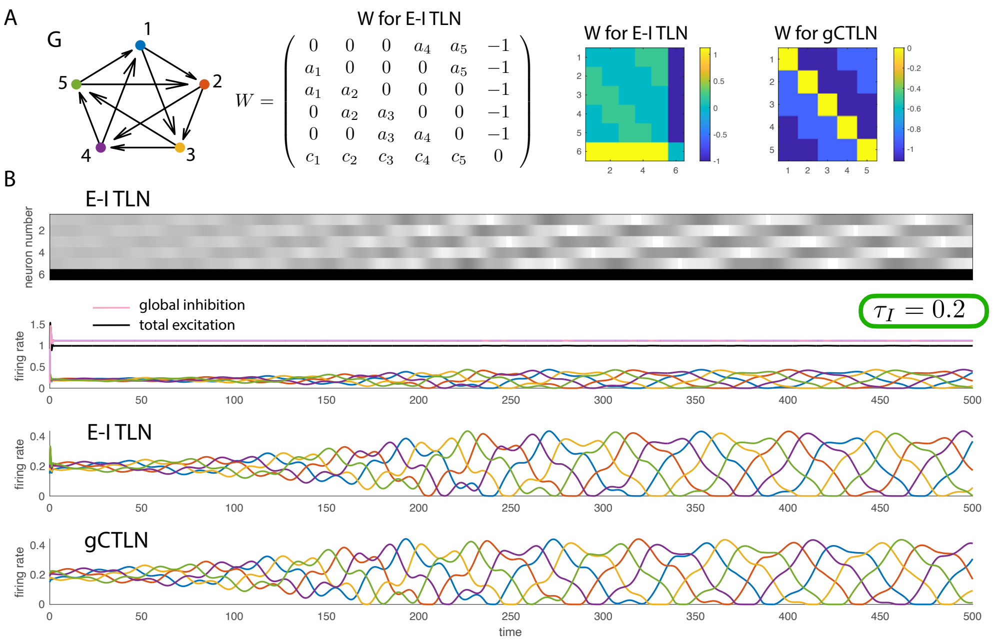

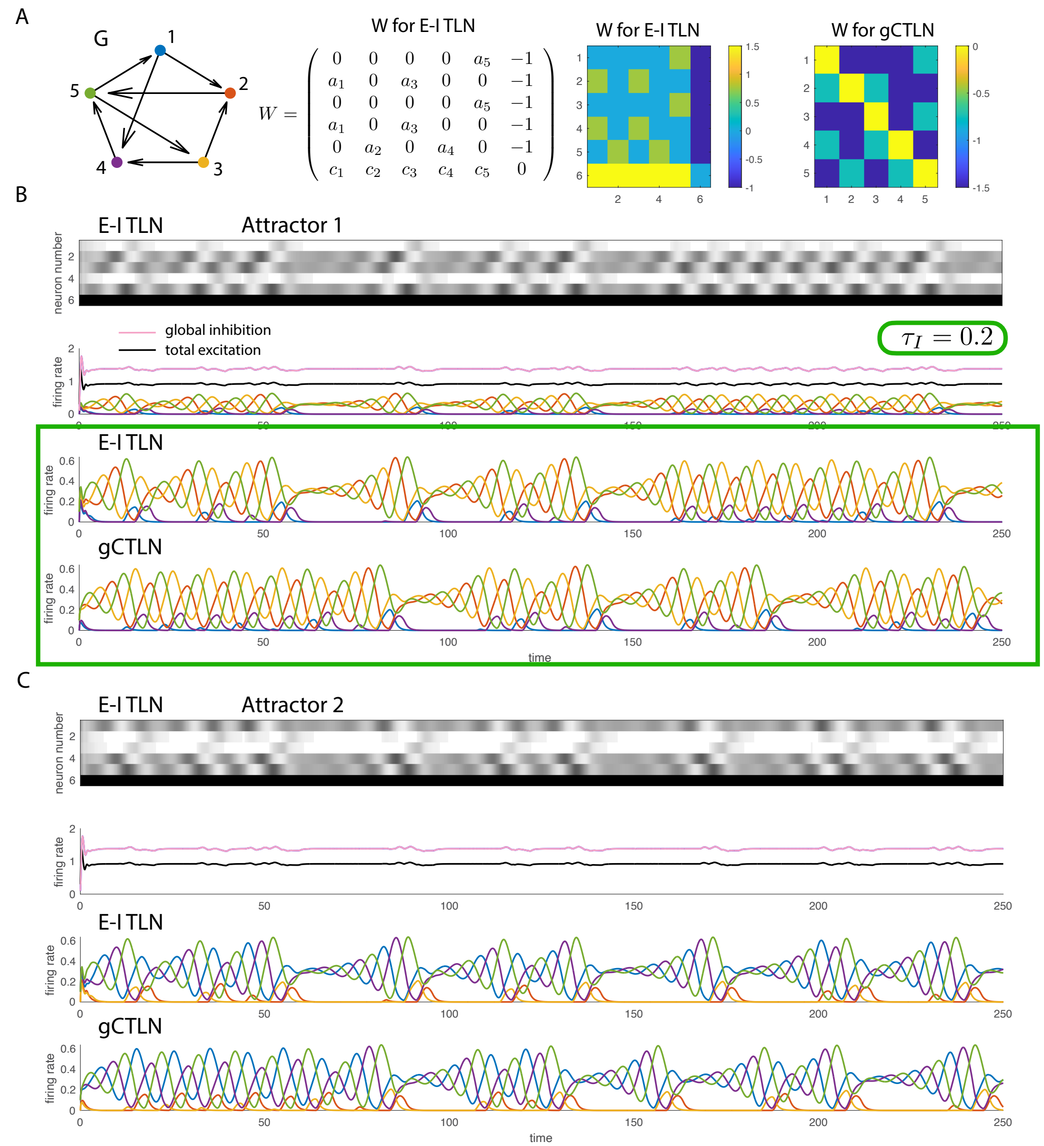

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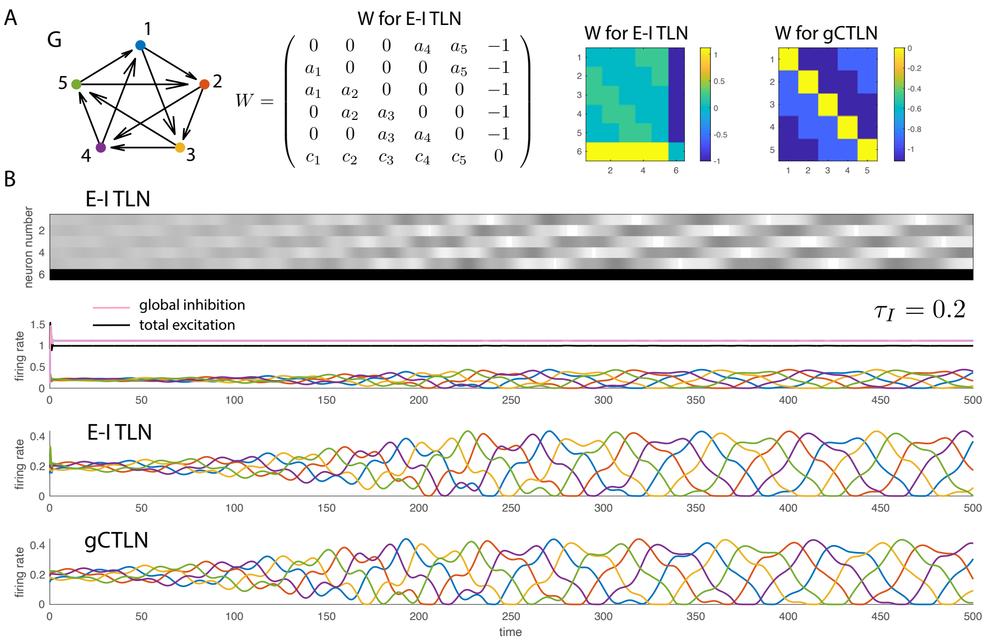
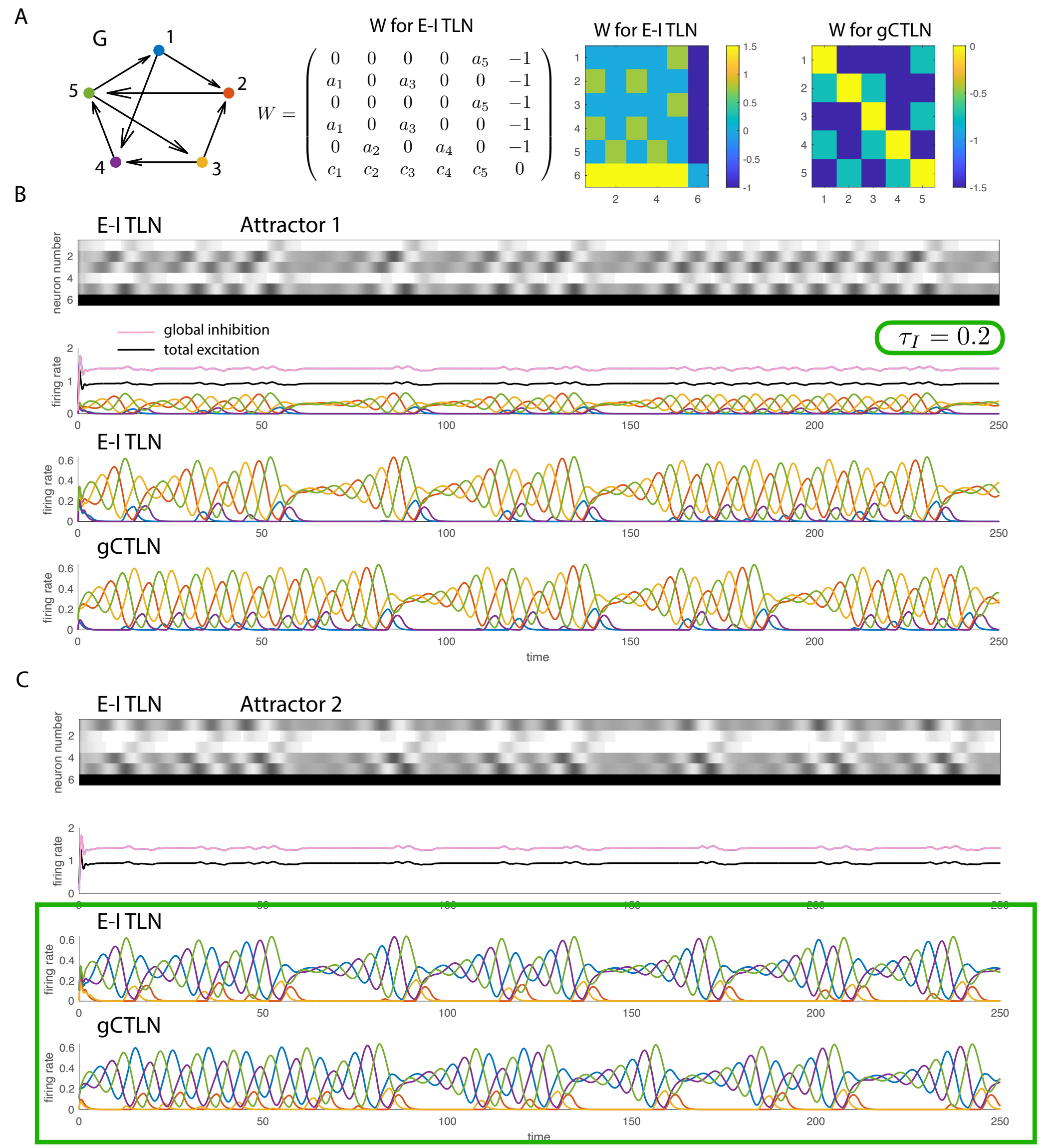
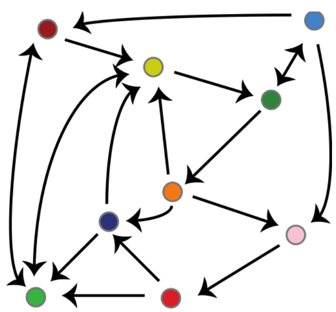
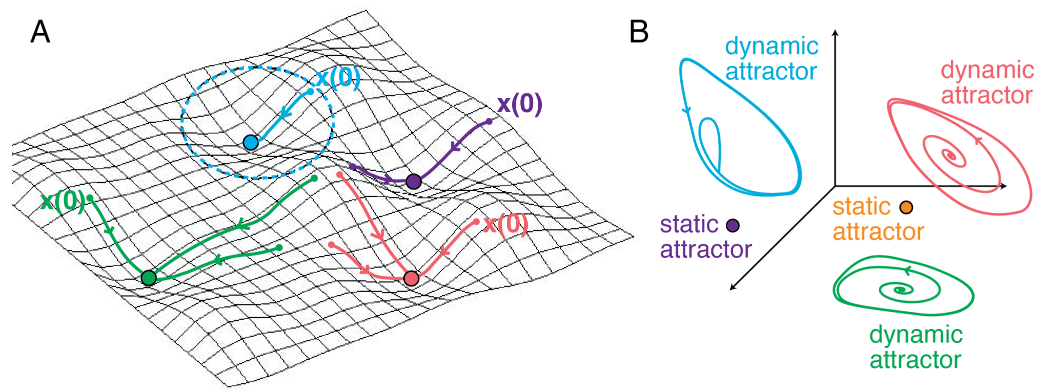




We had many mathematical results, called “graph rules” on CTLNs.

Now many of those results also apply to E-I TLNs built from graphs!





Curto & Morrison, 2023 (review paper): Graph rules for recurrent neural network dynamics

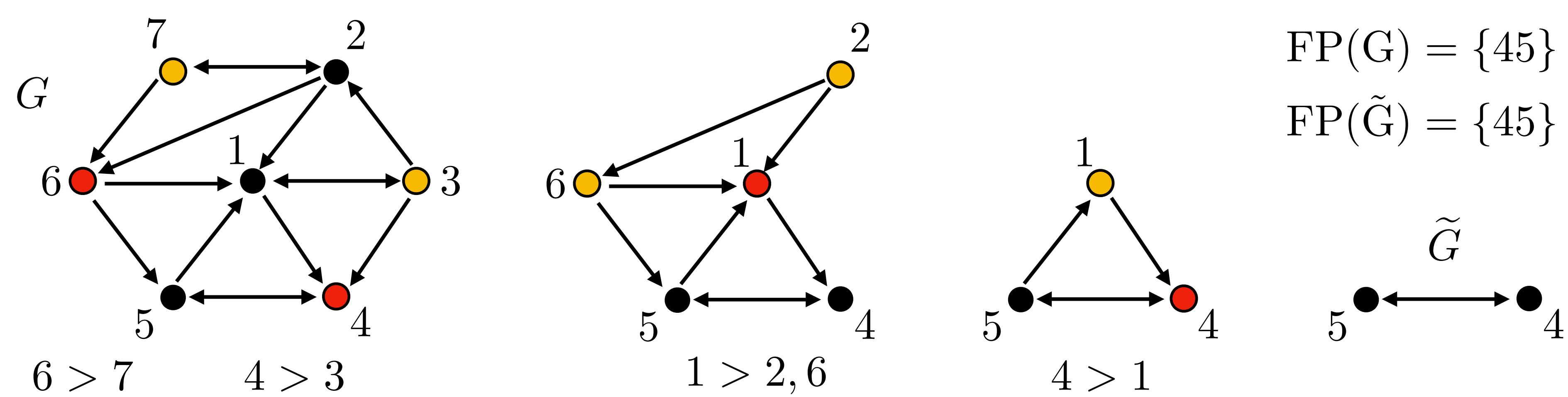
Domination Theorems

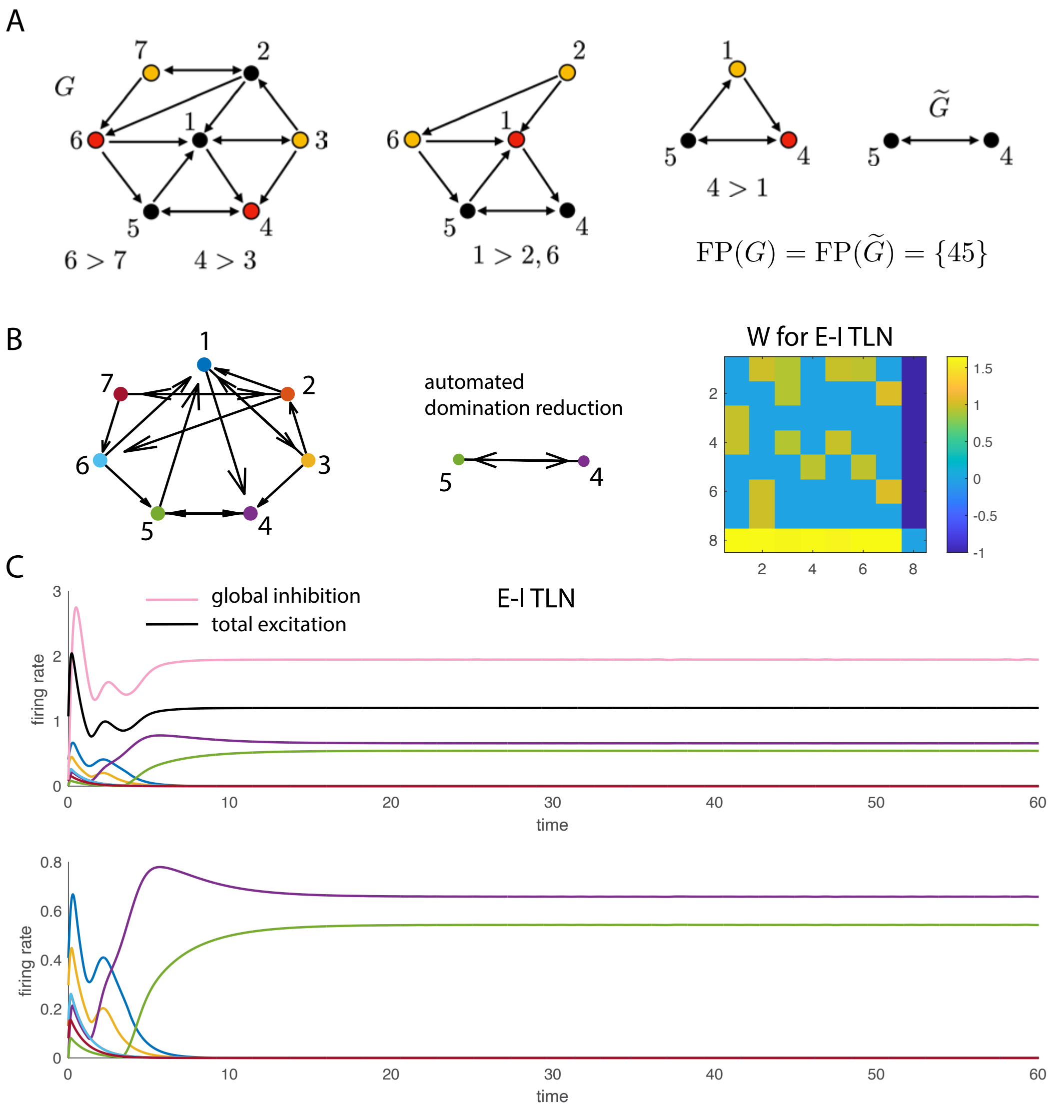
Theorem 1 (2024)
If j is a dominated node in G, then it drops out!
I.e., in any gCTLN, we have: $\mathrm{FP}(G) = \mathrm{FP}(G|_{[n]\setminus j})$

Theorem 2 (2024)
By iteratively removing dominated nodes, the final reduced graph G-tilde is unique. Moreover, $\mathrm{FP}(G) = \mathrm{FP}(\widetilde{G})$

Since E-I TLNs map to gCTLNs with the same fixed points, the domination theorems hold for E-I TLNs, too!

Example





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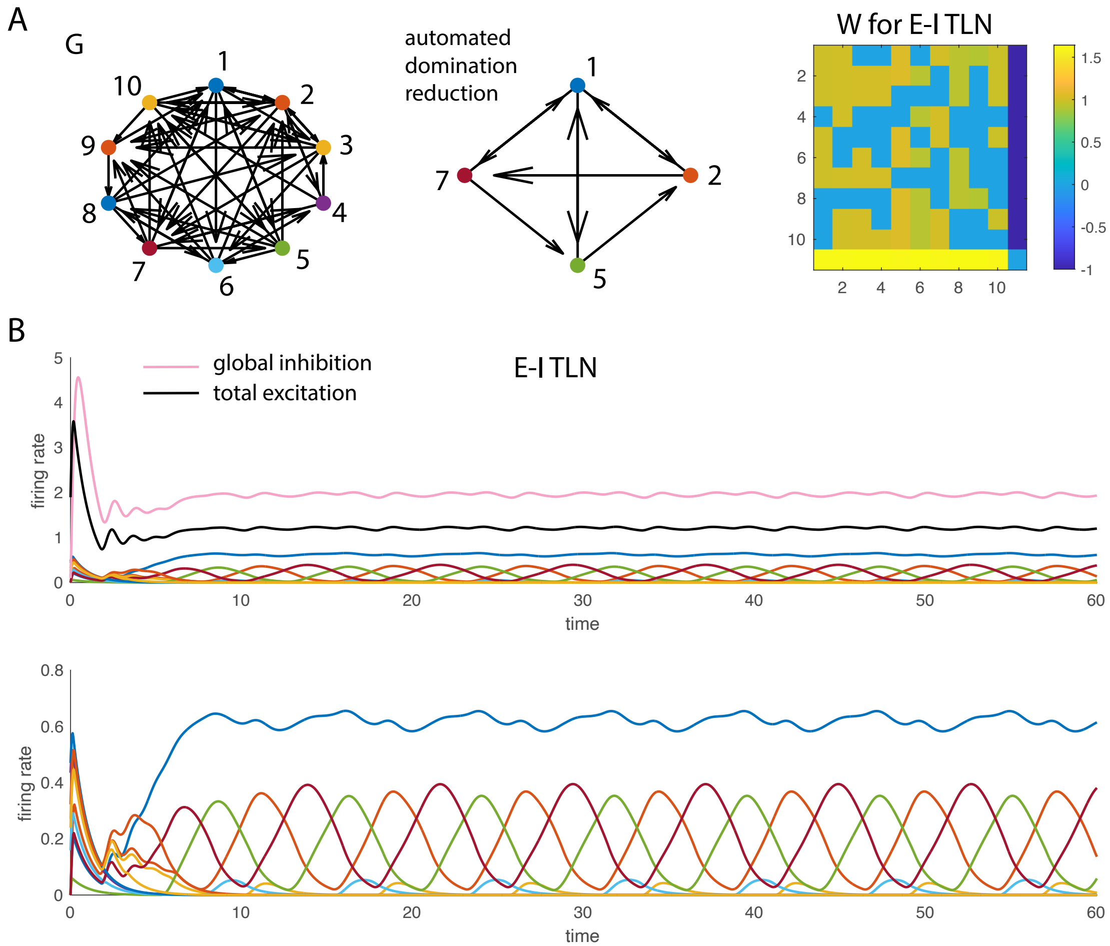

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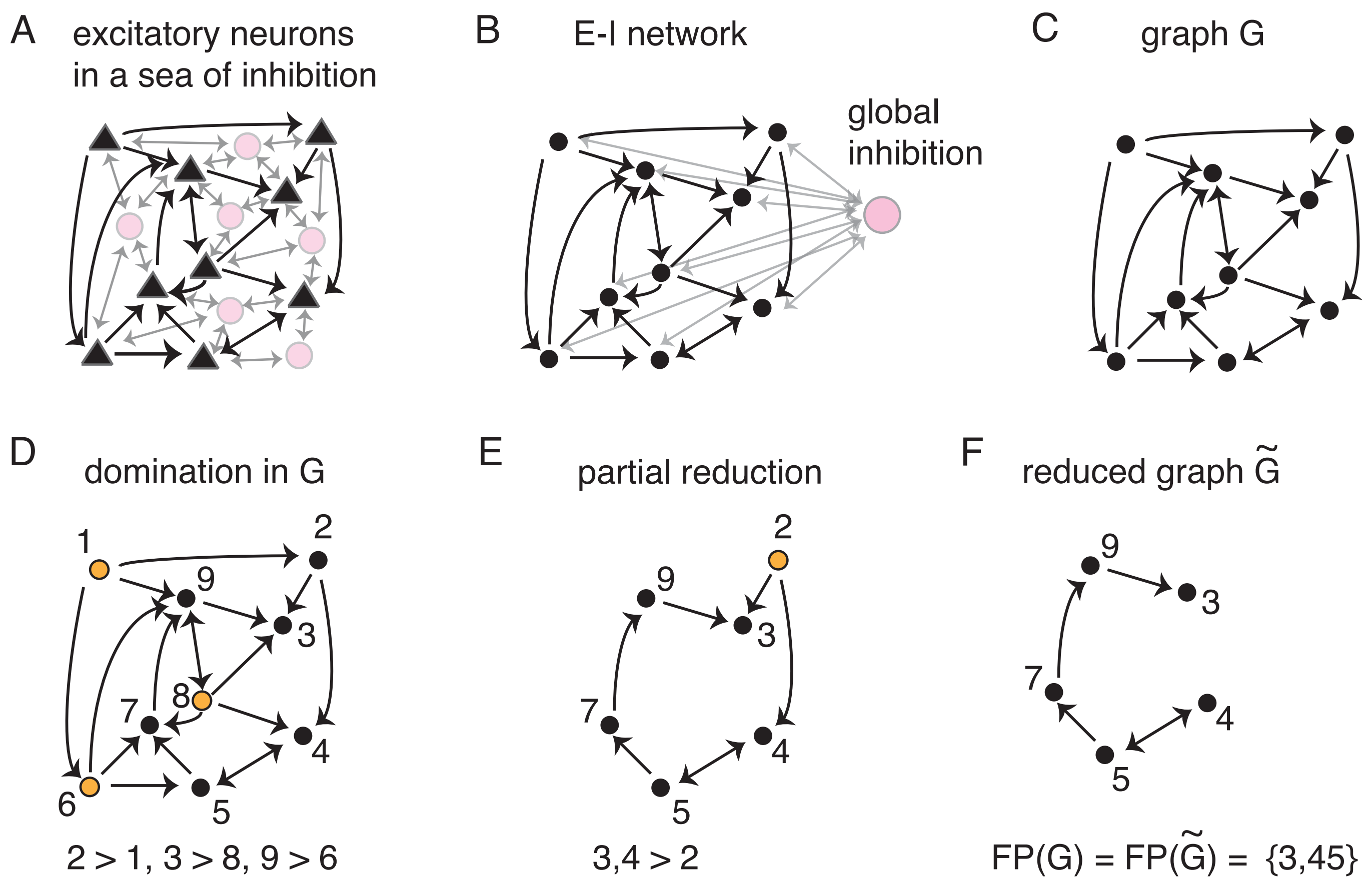
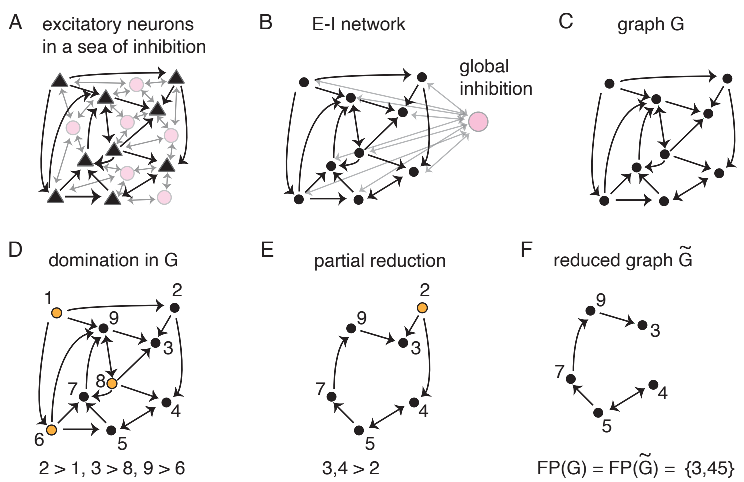

A
excitatory neurons in a sea of inhibition
B
E-I network
global inhibition
C
graph G
D
domination in G
1
2
9
3
7
8
4
6
5
2 > 1, 3 > 8, 9 > 6
E
partial reduction
2
9
3
7
4
5
3,4 > 2
F
reduced graph G̃
9
3
7
4
5
FP(G) = FP(G̃) = {3,45}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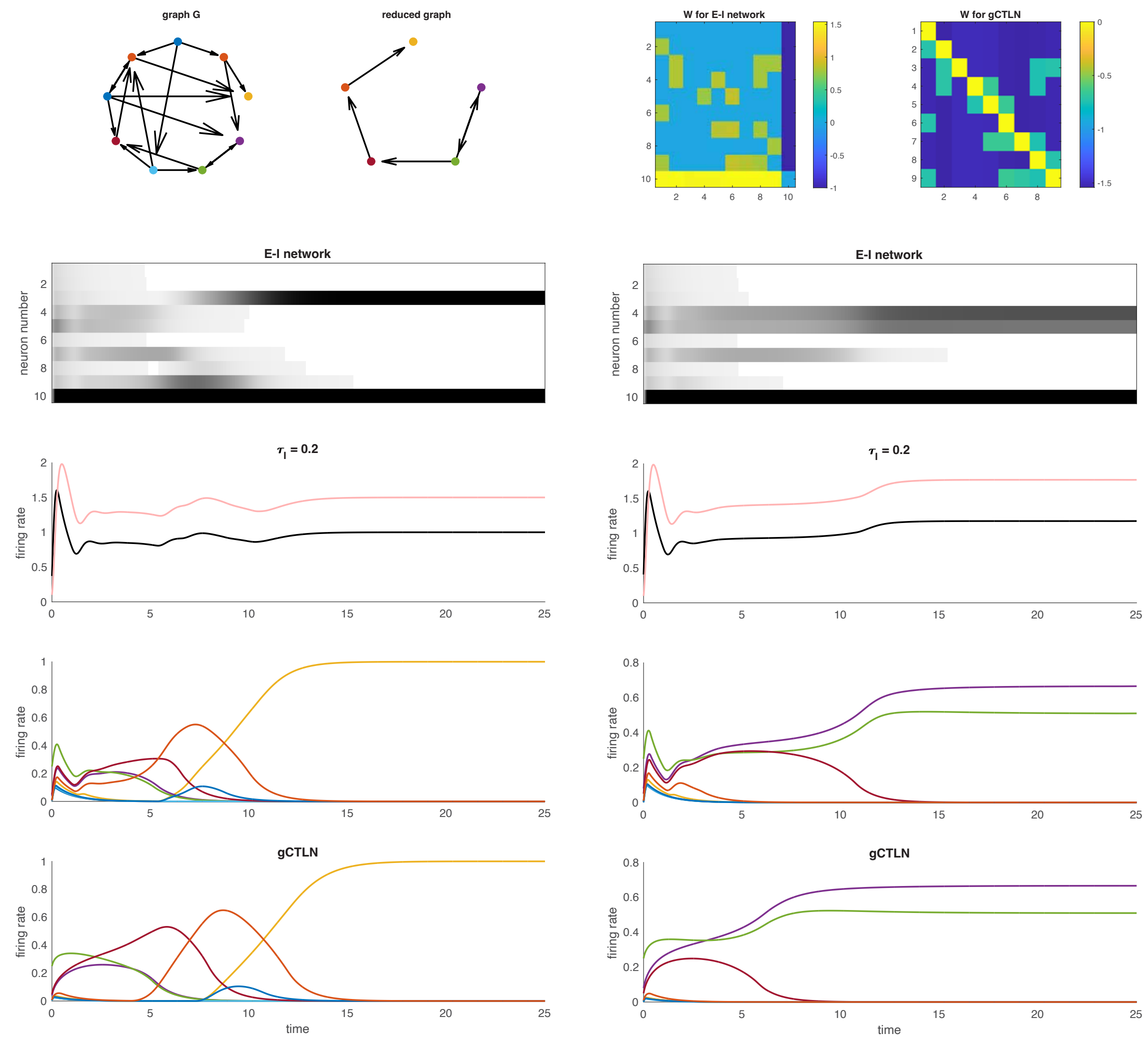
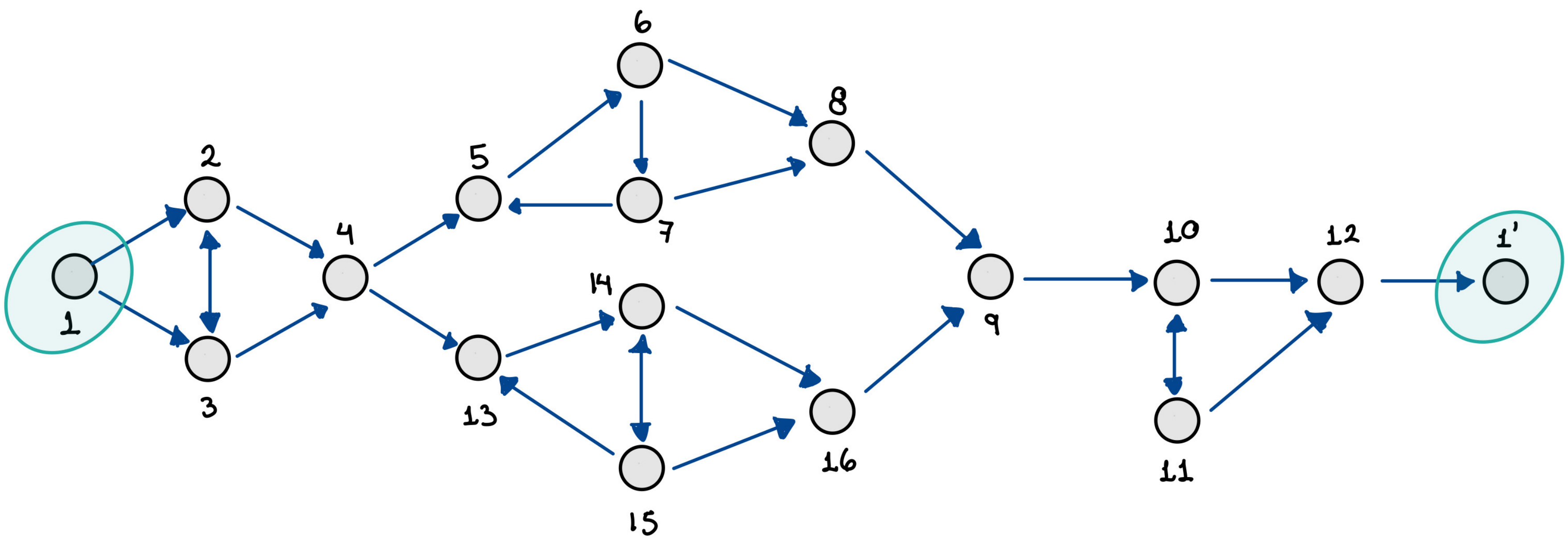
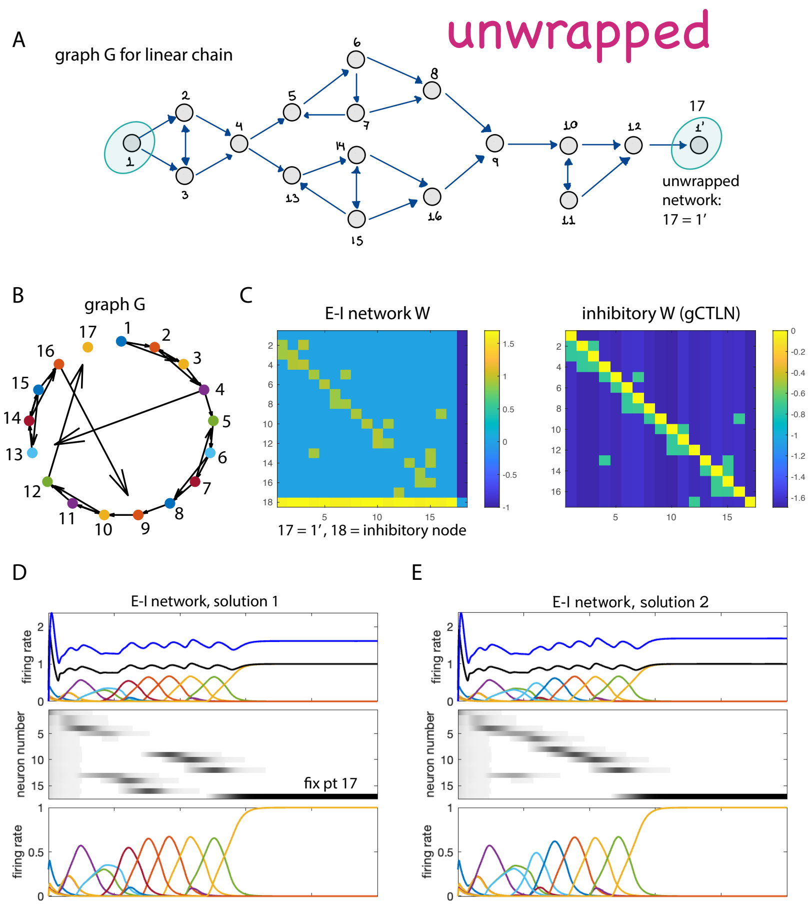

graph G
reduced graph
W for E-I network
W for gCTLN
E-I network
neuron number
τ_I = 0.2
firing rate
gCTLN
time

Cyclic chain example

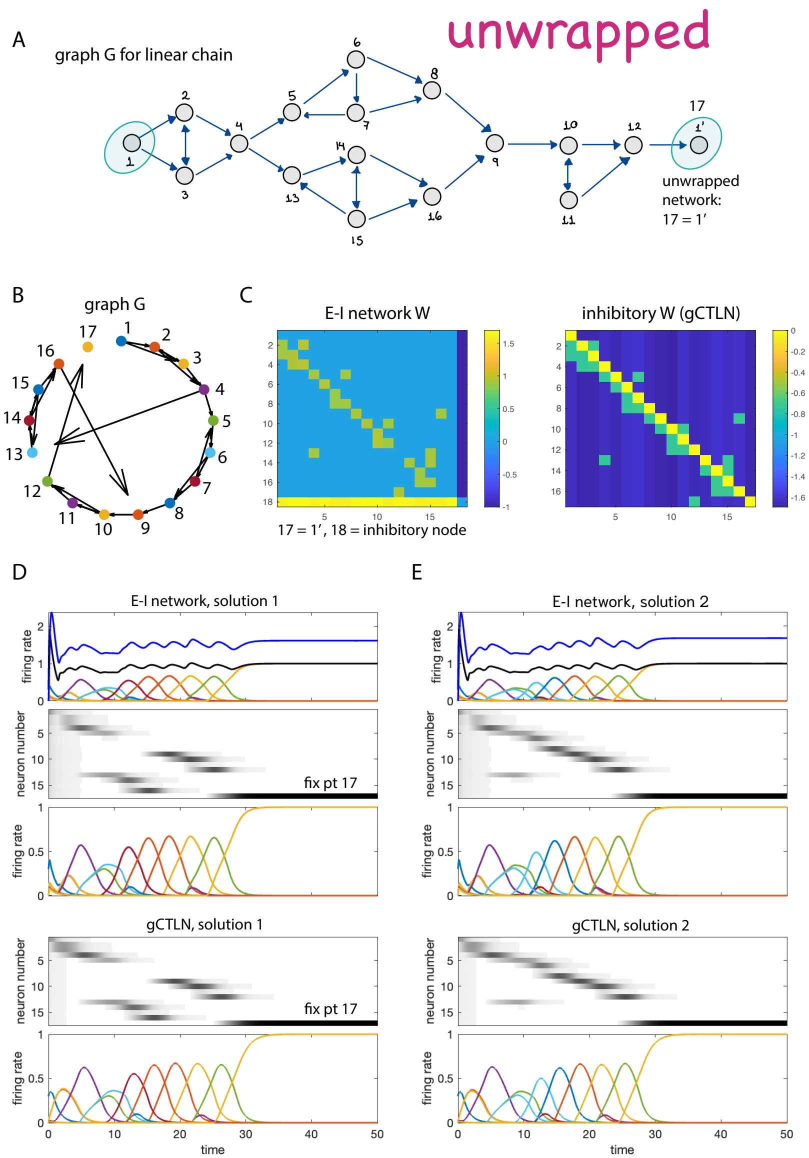


Identify 1 ≡ 1' at the end

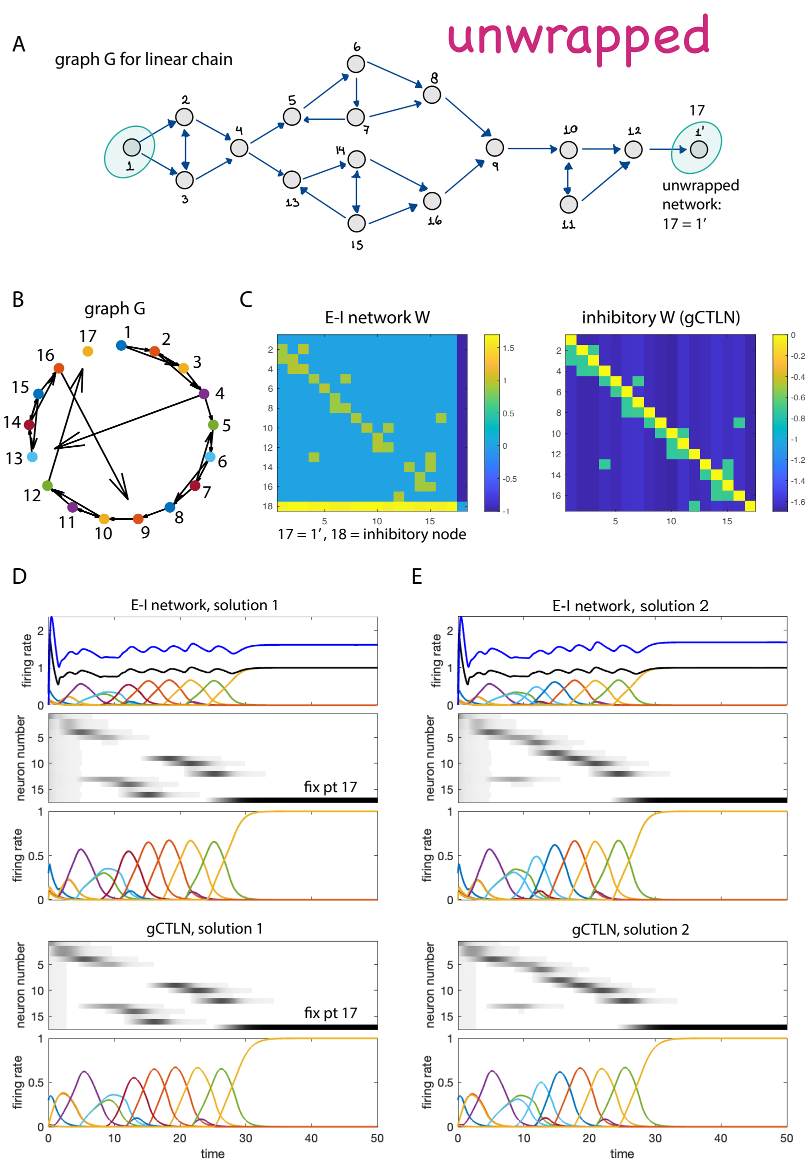
Domination reduction cannot be done, and the network activity will loop around.

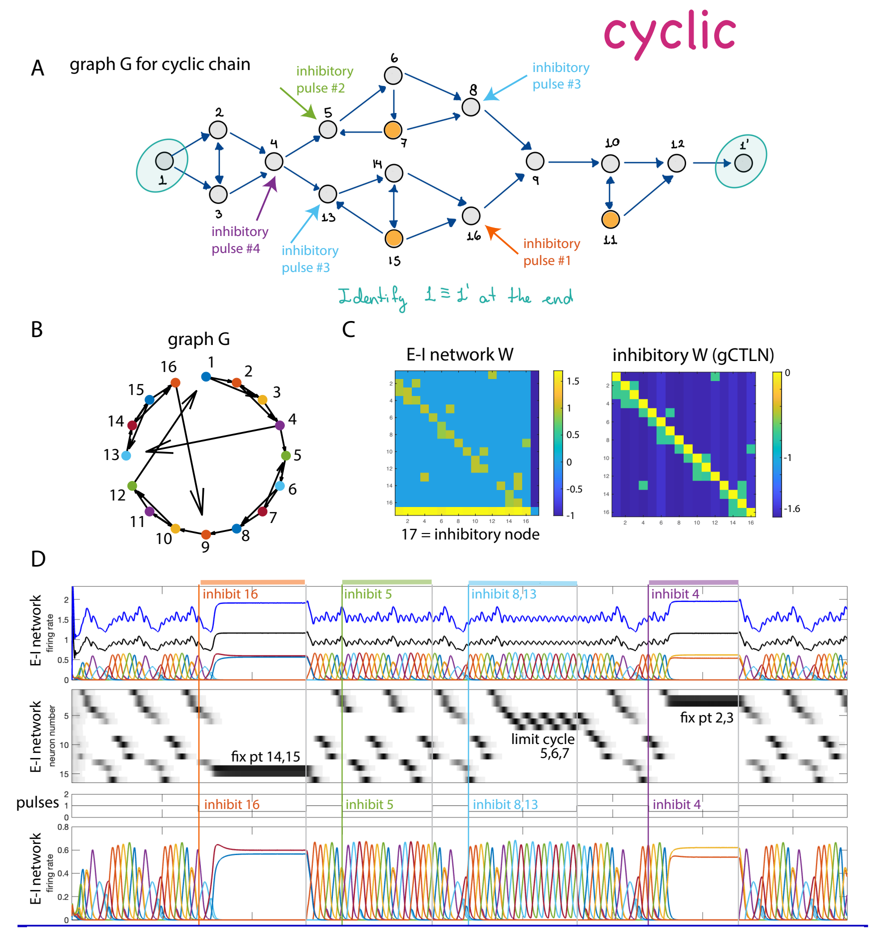


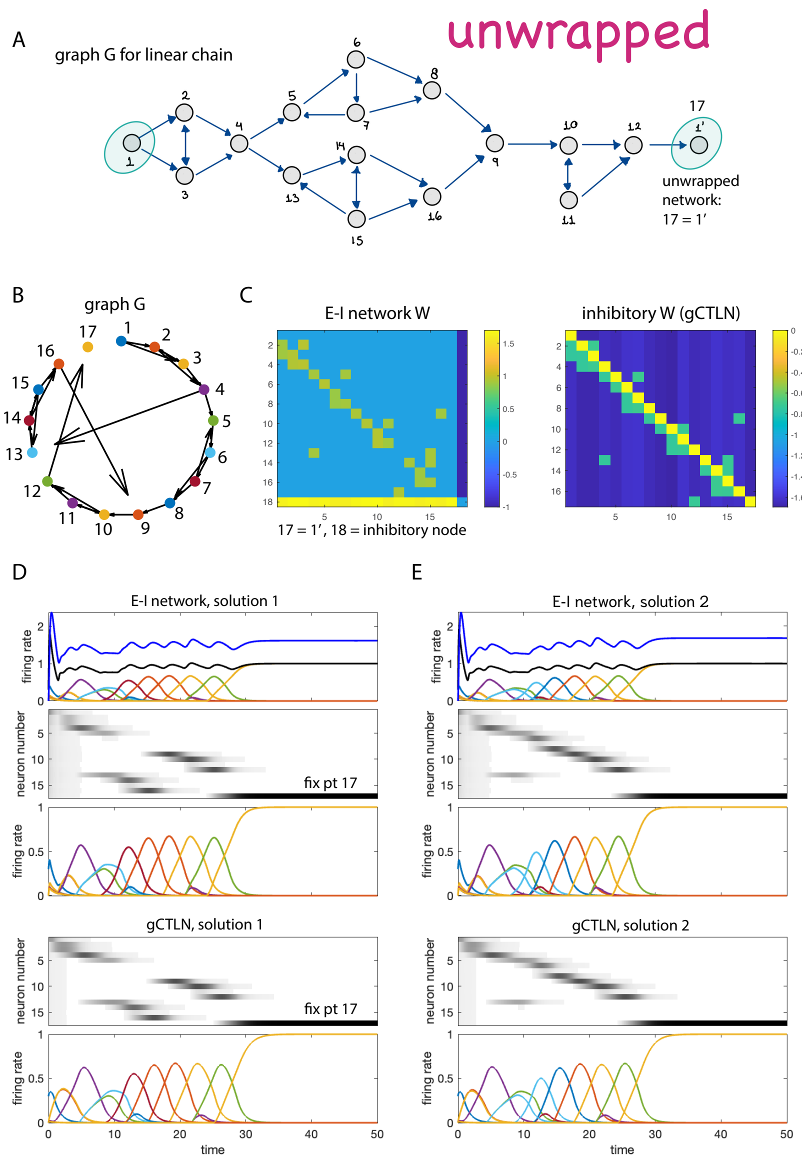
unwrapped
A
graph G for linear chain
17
unwrapped
network:
17 = 1′
B
graph G
C
E-I network W
17 = 1′, 18 = inhibitory node
inhibitory W (gCTLN)
D
E-I network, solution 1
firing rate
neuron number
fix pt 17
E
E-I network, solution 2



unwrapped
A
graph G for linear chain
unwrapped network: 17 = 1′
B
graph G
C
E-I network W
inhibitory W (gCTLN)
17 = 1′, 18 = inhibitory node
D
E-I network, solution 1
fix pt 17
gCTLN, solution 1
E
E-I network, solution 2
gCTLN, solution 2
firing rate
neuron number
time


unwrapped
A
graph G for linear chain
17
unwrapped network: 17 = 1′
B
graph G
C
E-I network W
inhibitory W (gCTLN)
17 = 1′, 18 = inhibitory node
D
E-I network, solution 1
fix pt 17
gCTLN, solution 1
E
E-I network, solution 2
gCTLN, solution 2
firing rate
neuron number
time


cyclic
A
graph G for cyclic chain
inhibitory pulse #2
inhibitory pulse #3
inhibitory pulse #4
inhibitory pulse #1
Identify 1 ≡ 1′ at the end
B
graph G
C
E-I network W
inhibitory W (gCTLN)
17 = inhibitory node
D
inhibit 16
inhibit 5
inhibit 8,13
inhibit 4
E-I network
firing rate
neuron number
fix pt 14,15
limit cycle 5,6,7
fix pt 2,3
pulses


unwrapped
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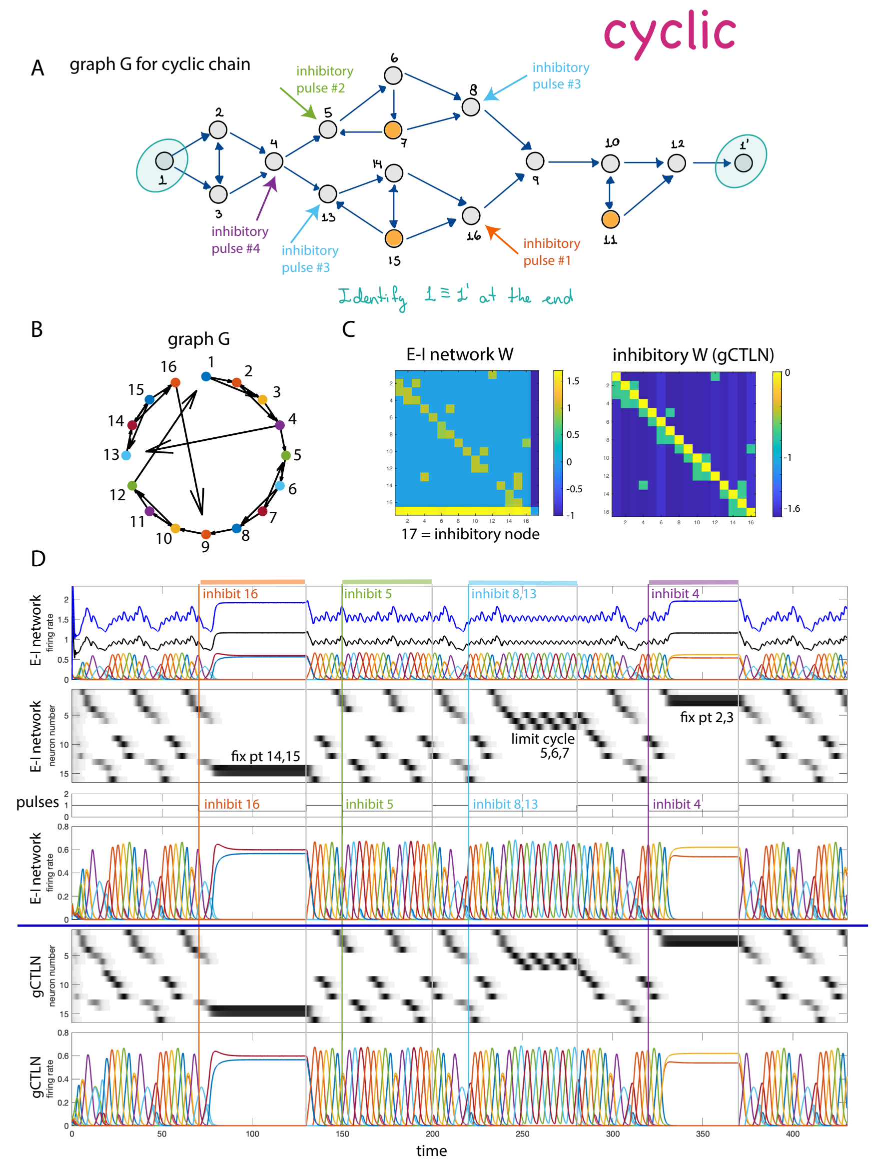
graph G for linear chain
unwrapped network:
17 = 1′
B
graph G
C
E-I network W
inhibitory W (gCTLN)
17 = 1′, 18 = inhibitory node
D
E-I network, solution 1
fix pt 17
gCTLN, solution 1
E
E-I network, solution 2
gCTLN, solution 2
firing rate
neuron number
time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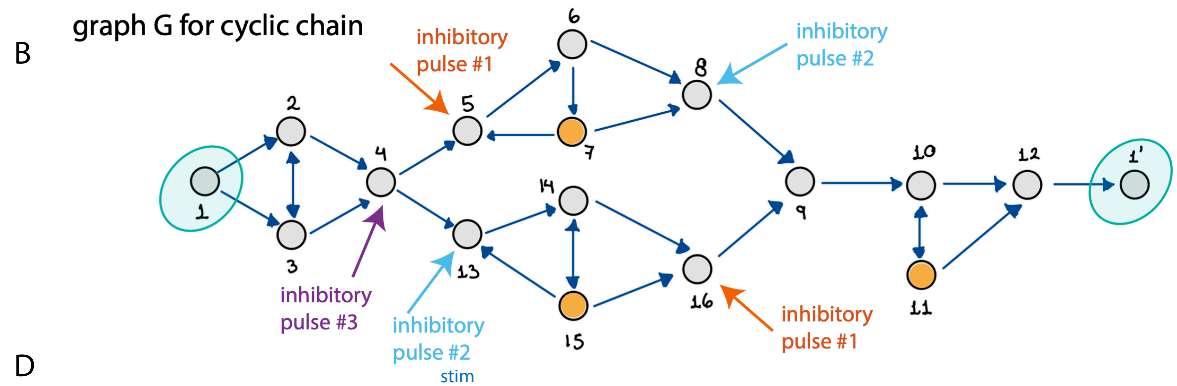
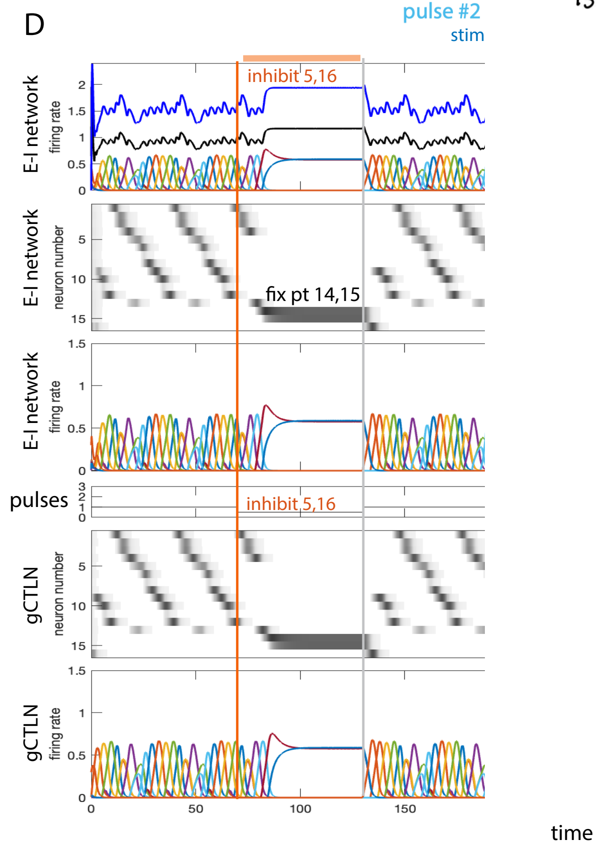
pulses
E-I network
gCTLN
firing rate
neuron number
time

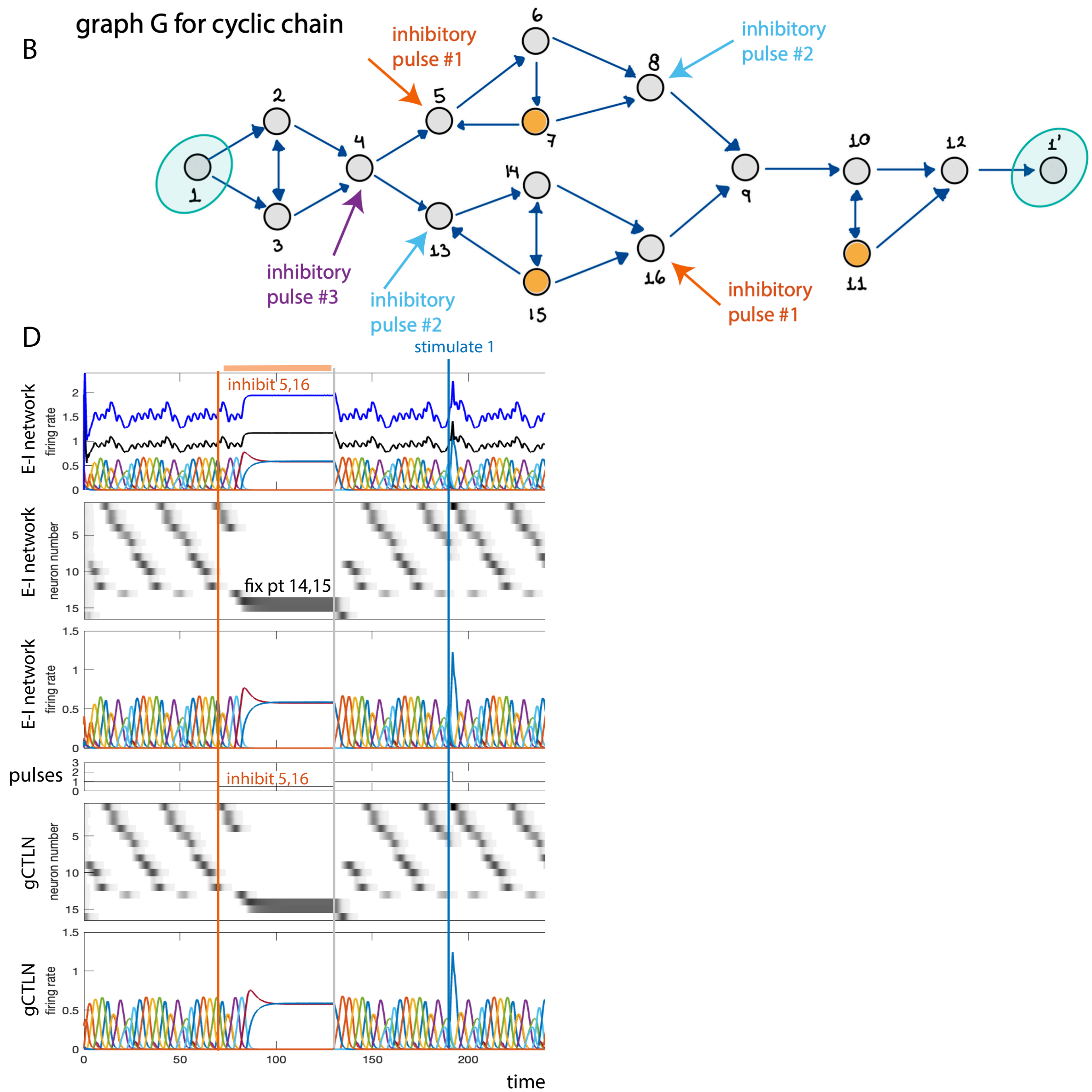
inhibitory pulses
= stop signs





inhibitory pulses
= stop signs

excitatory pulses
= teleportation



inhibitory pulses
= stop signs

excitatory pulses
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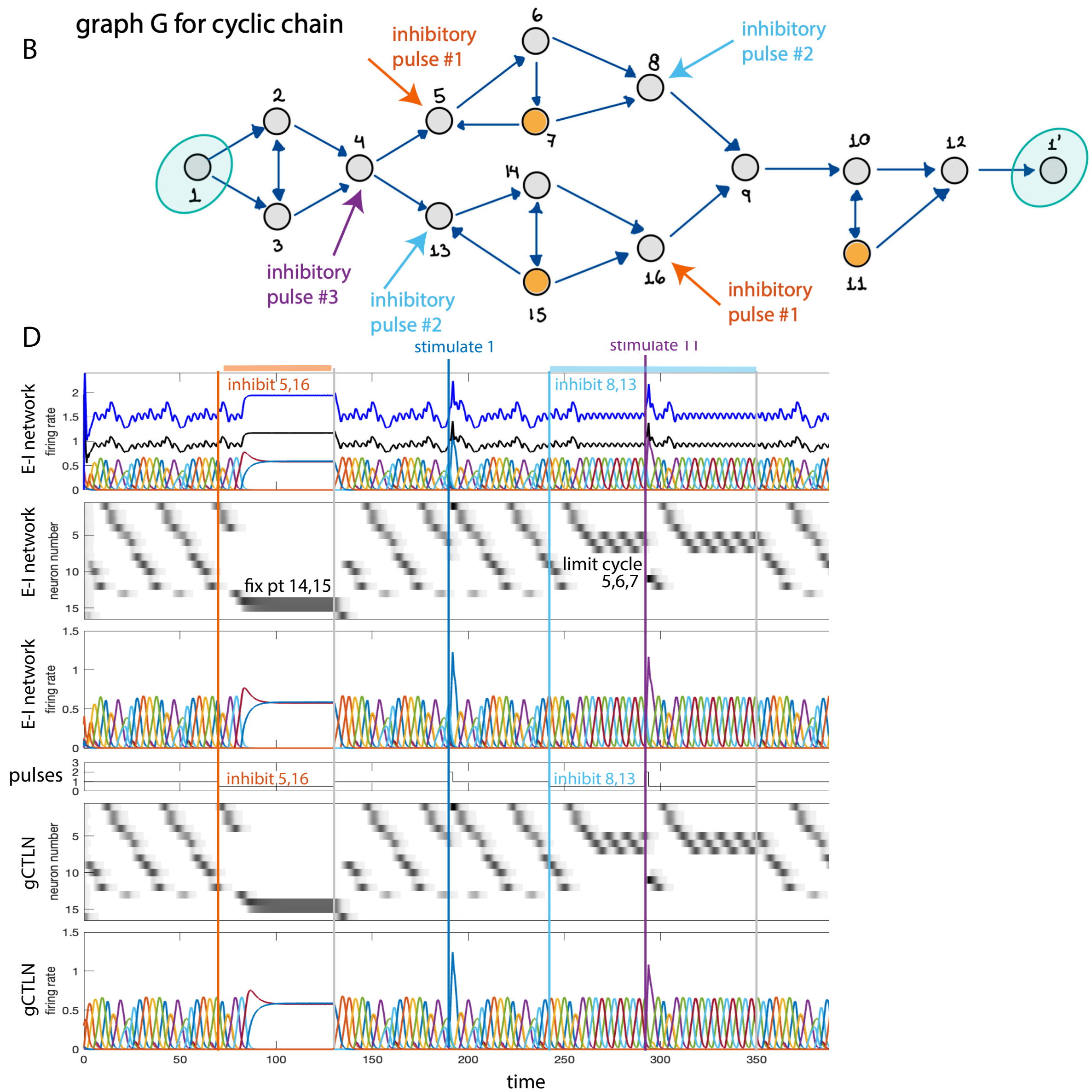

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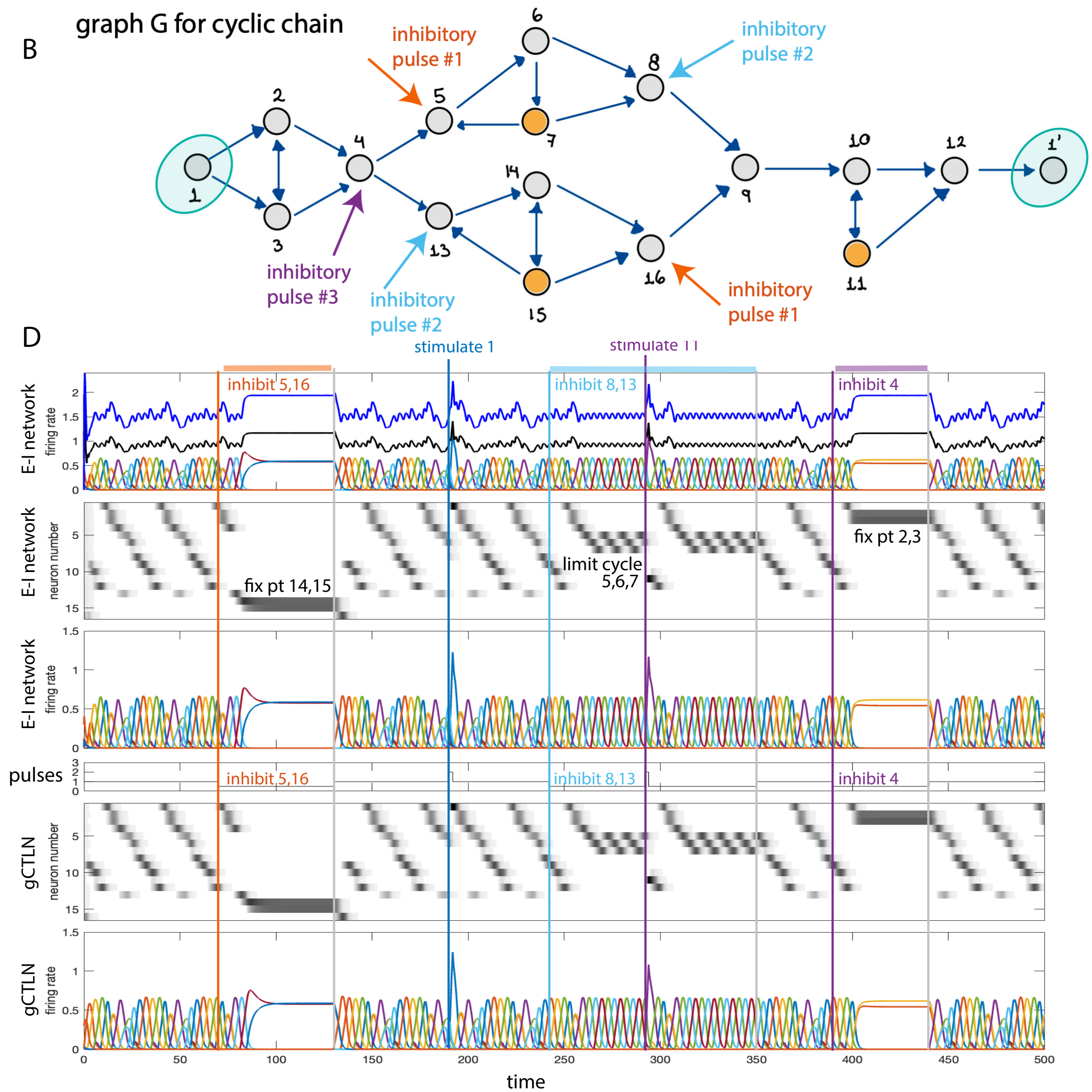


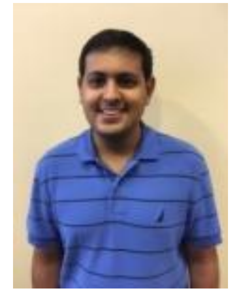
Thank you!


Katie Morrison

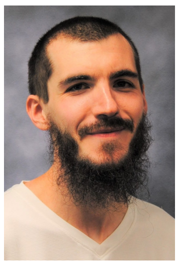

Caitlyn Parmelee


Chris Langdon


Nicole Sanderson


Safaan Sadiq


Jency (Yuchen) Jiang


Jesse Geneson


Caitlin Lienkaemper


Juliana Londoño Alvarez


Joaquín Castañeda Castro

grad student:
Zelong Li

Other Collaborators:


Vladimir Itskov


Anda Degeratu

Jordan Matelsky (also at Penn)
Patricia Rivlin
Michael Robinette
Erik Johnson
Brock Wester

Johns Hopkins University Applied Physics Laboratory,
Research & Exploratory Development Department

Domination - New Theorems - a word about the proofs

3. Proof of Theorem 1.5 Theorem 1

In order to prove Theorem 1.5, it will be useful to use the notation

$$y_i(x) = \sum_{j=1}^{n} W_{ij} x_j + b_i. \tag{3.1}$$

With this notation, the equations for a TLN (W, b) become:

$$\frac{dx_i}{dt} = -x_i + [y_i(x)]_+.$$

If x^* is a fixed point of (W, b), then $x_i^* = [y_i^*]_+$, where $y_i^* = y_i(x^*)$.

We can now prove the following technical lemma:

Lemma 3.2. *Let (W, b) be a TLN on n nodes and consider distinct $j, k \in [n]$. If $W_{ji} \leq W_{ki}$ for all $i \neq j, k$, and $b_j \leq b_k$, then for any fixed point x^* of (W, b) we have*

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Furthermore, if $W_{kj} > -1$ and $W_{jk} \leq -1$, then

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Proof. Suppose x^* is a fixed point of (W, b) with support $\sigma \subseteq [n]$. Then, recalling that $W_{jj} = W_{kk} = 0$ and that $x_i^* = 0$ for all $i \notin \sigma$, from equation (3.1) we obtain:

$$y_j^* - W_{jk} x_k^* = \sum_{i \in \sigma \setminus \{j,k\}} W_{ji} x_i^* + b_j,$$

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Lemma 3.5. *Suppose j is a dominated node in G. Then, for any associated gCTLN, $y_j^* \leq 0$ at every fixed point x^* (no matter the support).*

Proof. Suppose j is a dominated node in G. Then, there exists $k \in [n]$ such that $j \to k$, $k \not\to j$, and satisfying $i \to k$ whenever $i \to j$. Translating these conditions to an associated gCTLN, with weight matrix given as in equation (1.3), we can see that $W_{kj} > -1$, $W_{jk} < -1$, and $W_{ji} \leq W_{ki}$ for all $i \neq j, k$. Moreover, since $b_j = b_k = \theta$, we also satisfy $b_j \leq b_k$. We are thus precisely in the setting of the second part of Lemma 3.2, and we can conclude that $y_j^* \leq 0$ at any fixed x^* of the gCTLN. □

need some more lemmas...

Lemma 3.6. *Let G be a graph with vertex set $[n]$. For any gCTLN on G,*

$$\sigma \in \text{FP}(G) \iff \sigma \in \text{FP}(G|_\omega) \text{ for all } \omega \text{ such that } \sigma \subseteq \omega \subseteq [n]$$
$$\iff \sigma \in \text{FP}(G|_\sigma) \text{ and } \sigma \in \text{FP}(G|_{\sigma \cup \ell}) \text{ for all } \ell \notin \sigma.$$

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Proof of Theorem 1

Proof of Theorem 1.5. Suppose j is a dominated node in G, and let (W, b) be an associated gCTLN. By Lemma 3.5, we know that $y_j^* \leq 0$ at every fixed point (W, b). It follows that $j \notin \sigma$ for all $\sigma \in \text{FP}(G)$. Hence,

$$\text{FP}(G) \subseteq \text{FP}(G|_{[n]\setminus j}).$$

It remains to show that $\text{FP}(G|_{[n]\setminus j}) \subseteq \text{FP}(G)$. By Lemma 3.6, this is equivalent to showing that for each $\sigma \in \text{FP}(G|_{[n]\setminus j})$, $\sigma \in \text{FP}(G|_{\sigma \cup j})$.

Suppose $\sigma \in \text{FP}(G|_{[n]\setminus j})$, with corresponding fixed point x^*. In this setting, we are not guaranteed that $y_j^* = y_j(x^*) \leq 0$, as x^* is not necessarily a fixed point of the full network. To see whether $\sigma \in \text{FP}(G|_{\sigma \cup j})$, if suffices to check the "off"-neuron condition for j: that is, we need to check if $y_j^* \leq 0$ when evaluating (3.1) at x^*.

Recall now that there exists a $k \in [n] \setminus j$ such that k graphically dominates j. It is also useful to evaluate y_k^* at x^*. Following the beginning of the proof of Lemma 3.2, we see that simply from the fact that $\text{supp}(x^*) = \sigma$, we obtain

$$y_j^* + W_{kj}x_j^* \leq y_k^* + W_{jk}x_k^*.$$

However, we cannot assume $x_j^* = [y_j^*]_+$, since we are not necessarily at a fixed point of the full network (W, b). We know only that $x_j^* = 0$ and $x_k^* = [y_k^*]_+$, as the fixed point conditions are satisfied in the subnetwork $(W_{[n]\setminus j}, b_{[n]\setminus j})$ that includes k. This yields,

$$y_j^* \leq y_k^*(1 + W_{jk}) \leq 0,$$

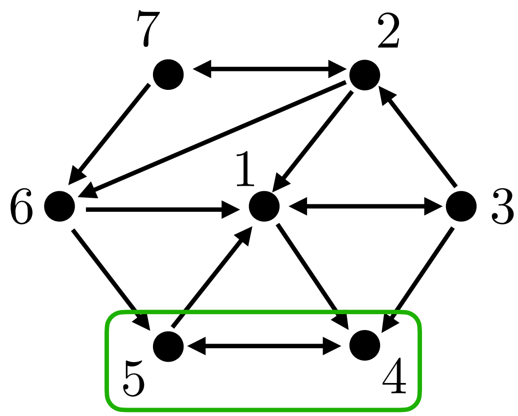
where the second inequality stems from the fact that $W_{jk} < -1$. So, as it turns out, we see that $y_j^* \leq 0$ not only for fixed points of (W, b), but also for fixed points from the subnetwork $(W_{[n]\setminus j}, b_{[n]\setminus j})$. We can thus conclude that $\text{FP}(G|_{[n]\setminus j}) \subseteq \text{FP}(G)$, completing the proof. □

Can domination be useful for connectome analysis?

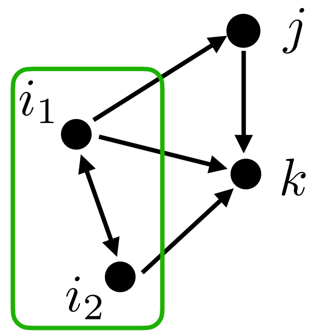
Every graph has a unique domination reduction: $G \longrightarrow \widetilde{G}$

Two graphs with the same reduction are in the same domination equivalence class.

$G \longrightarrow \widetilde{G}$



$H \longrightarrow \widetilde{H}$



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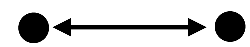
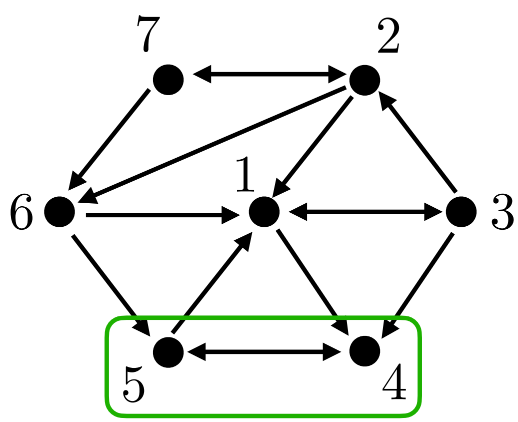


Can domination be useful for connectome analysis?

Every graph has a unique domination reduction: $G \longrightarrow \widetilde{G}$

Two graphs with the same reduction are in the same domination equivalence class.

$G \longrightarrow \widetilde{G}$



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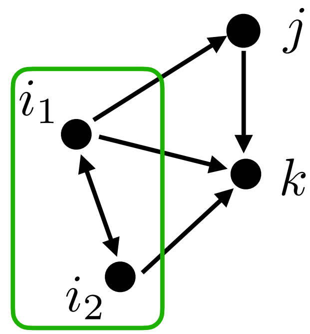

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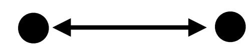


1. Are overrepresented graphical motifs more likely to be reducible or irreducible?

2. Which motifs are domination-equivalent?

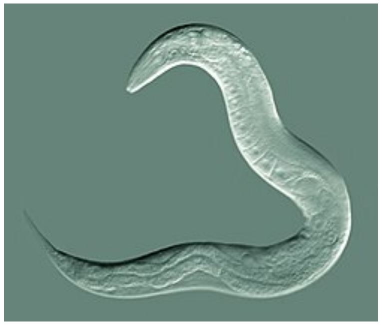
3. What about larger portions of the connectome: do they reduce via domination?

Very preliminary analysis

Graph motifs team at JHU

Jordan Matelsky (also at Penn)
Patricia Rivlin
Michael Robinette
Erik Johnson
Brock Wester

Johns Hopkins University Applied Physics Laboratory,
Research & Exploratory Development Department



C. elegans E-E network:

G has143 nodes

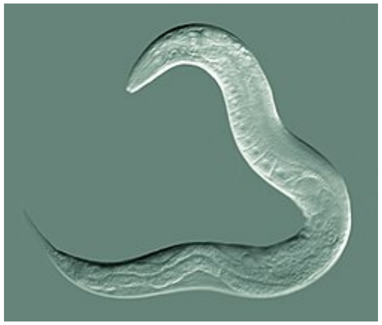
reduced G: 104 nodes

Joaquín Castañeda Castro

We first strip out everything but chemical synapses, then tag neurons by their small-molecule neurotransmitters—acetylcholine/glutamate as excitatory, GABA as inhibitory—next we grab the induced subgraph of neurons that fire ACh/Glu but no GABA. That's our 'excitatory' network. And yes—it's just a conservative, transmitter-based proxy for valence; real C. elegans synaptic polarity is far messier (receptors, modulators, co-transmission, gap junctions, etc.) All blame goes to Jordan Matelsky, Carina did nothing wrong.

Very preliminary analysis

Is a reduction from 143 -> 104 nodes common or rare in a random graph with matching edge probability?



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Very preliminary analysis

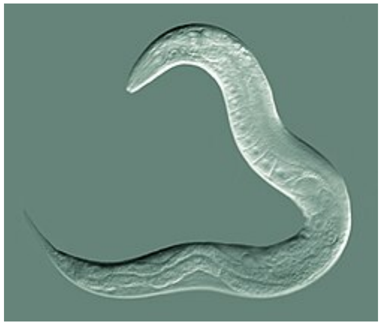
Is a reduction from 143 -> 104 nodes common or rare in a random graph with matching edge probability?

1 million E-R random graphs with matching p = 0.054

Distribution of domination reductions:

- 143 nodes: 782,590
- 142 nodes: 189,951
- 141 nodes: 24,951
- 140 nodes: 2,307
- 139 nodes: 185
- 138 nodes: 15
- 137 nodes: 1

VERY RARE!!



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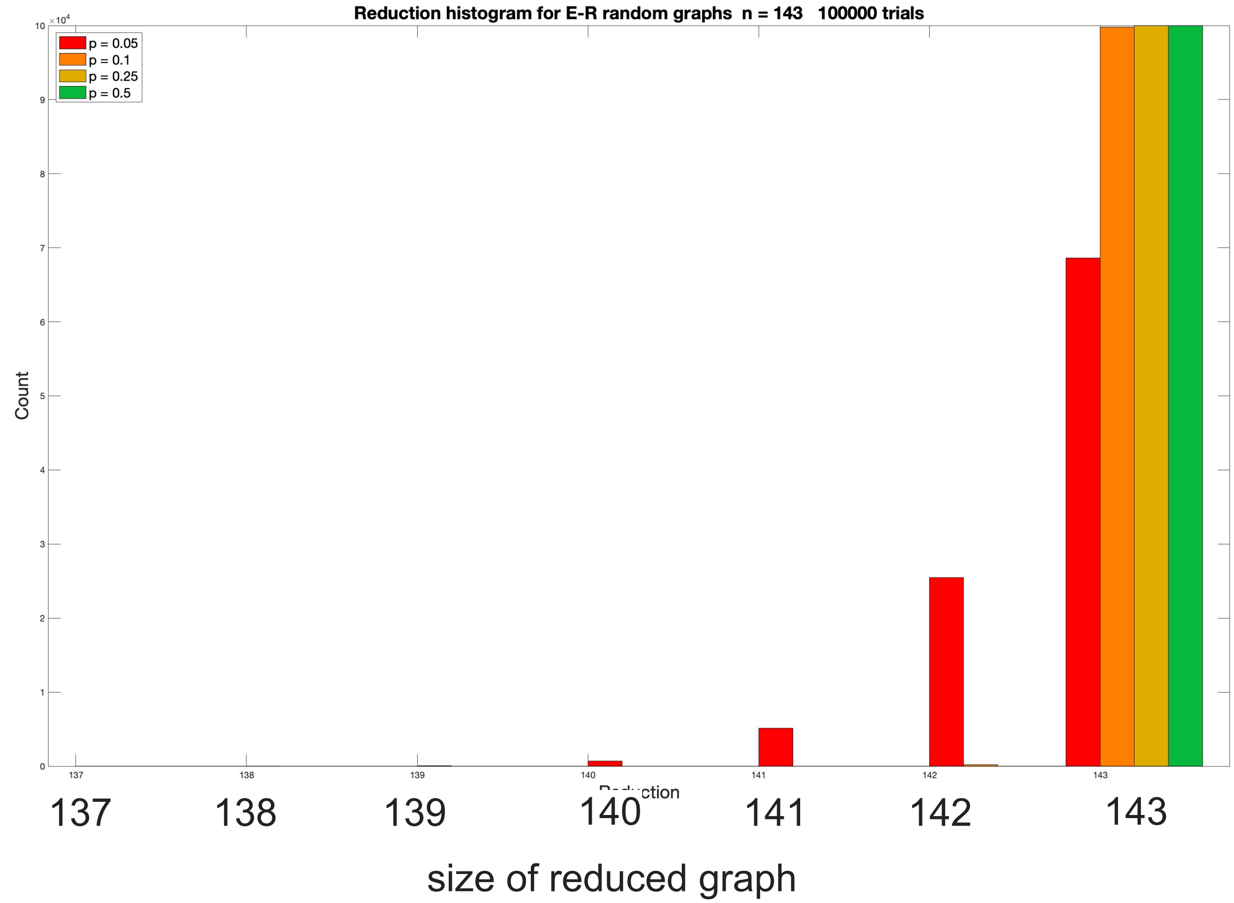
Reduction sizes of E-R random graphs of size n=143 with p = 0.05, 0.1, 0.25, 0.5



Back to our motivating questions and ideas:

1. How does connectivity shape dynamics?

2. The relationship between connectivity and neural activity depends on the dynamical system you associate to the connectome.

3. By studying neuroscience-inspired (nonlinear!) dynamical systems on graphs, we can generate hypotheses about the dynamic meaning/role of various network motifs.

Domination is a graph property that comes out of the nonlinear dynamics, it is not something that graph theorists or network scientists were already paying attention to.